Four Applications of Inverse Function Theorem

S. Kumaresan
School of Math. and Stat.
University of Hyderabad
Hyderabad 500046
kumaresa@gmail.com

Abstract

In this article, we prove four results which are typical applications of the inverse function theorem. The first is, of course, the implicit function theorem. The students should draw pictures (as was done by me in the calss) for all of the proofs so as to see the idea of the proof more clearly. Learn these proofs well. If you bring to my attention any errors of omission or commission, I may include pictures for your benefit later!

Theorem 1 (Implicit function theorem)**.** *Let $\Omega \subset \mathbb{R}^{n+k} = \mathbb{R}^n \times \mathbb{R}^k$ be open. Let $f\colon \Omega \to \mathbb{R}^k$ be C^1. Assume that for some $(a,b) \in \Omega$ where $a \in \mathbb{R}^n$ and $b \in \mathbb{R}^k$ we have*

1. $f(a,b) = 0$.
2. $D_2 f(a,b)$ *is nonsingular.*

Then there exists a neighbourhood Ω' of (a,b) in $\mathbb{R}^n \times \mathbb{R}^k$, an open set $U \subset \mathbb{R}^n$ containing a and a C^1-map g on U such that

i. $D_2 f(x,y)$ *is nonsingular for all* $(x,y) \in \Omega'$,
ii. $\{(x,y) \in \Omega' : f(x,y) = 0\} = \{(x, g(x)) : x \in U\}$.

Proof. Let $F\colon \Omega \to \mathbb{R}^n \times \mathbb{R}^k$ be defined as follows: $F(x,y) = (x, f(x,y))$. Then F is C^1 and the derivative $DF(a,b)$ can be written in the matrix form

$$\begin{pmatrix} I_{n\times n} & 0 \\ D_1 f(a,b) & D_2 f(a,b) \end{pmatrix}.$$

This is clearly nonsingular.

Hence by the inverse function theorem, there exists a neighbourhood Ω' of (a,b) in Ω such that $F(\Omega')$ is a neighbourhood of $F(a,b) = (a,0)$ in $\mathbb{R}^n \times \mathbb{R}^k$. Let p_X and p_Y denote the projections onto $\mathbb{R}^n$ and $\mathbb{R}^k$ respectively. Let

$$U := \{x \in X : (x,0) \in F(\Omega')\}.$$

Since $F(\Omega')$ is open, so is U. Consider $g(x) := p_Y \circ F^{-1}(x,0)$ for $x \in U$. Clearly, g is C^1 on U. Also, if $(x,y) \in \Omega'$, then $F(x,y) = 0$ iff $x \in U$ and $F(x,y) = (x,0)$. Applying F^{-1} to both sides, we get $F(x,y) = (x,0)$ iff $(x,y) = F^{-1}(x,0) = (x, g(x))$. This proves (ii) and completes the proof of the theorem. □

Theorem 2. *Let $S \subset \mathbb{R}^n$ be a surface. Let (V, φ, U) be a patch (or chart) in S. Let $W \subset \mathbb{R}^m$ be open and $F\colon W \to \mathbb{R}^n$ be smooth. Assume that $F(W) \subset U$. Then the map $\varphi^{-1} \circ F\colon W \to V$ is smooth.*

Proof. Note that while we know that $\varphi\colon V \to \mathbb{R}^n$ is smooth, since the domain of the inverse φ^{-1} is U, which is not an open set in $\mathbb{R}^n$, it makes no sense (so far) to talk of the differentiability of φ^{-1}.

Fix $w \in W$. Let $p = F(W)$ ad let $q \in V$ be such that $\varphi(q) = p$. We write $\varphi(x, u) = x(u, v) = (x_1(u, v), \ldots, x_n(u, v))$. Since $D\varphi(q)$ has rank 2, we may assume without loss of generality that the submatrix

$$\begin{pmatrix} \frac{\partial x_1}{\partial u} & \frac{\partial x_1}{\partial v} \\ \frac{\partial x_2}{\partial u} & \frac{\partial x_2}{\partial v} \end{pmatrix} (q)$$

has nonzero determinant.

Draw pictures as I did in the class now to understand the rest of the proof. We extend φ to $\Phi\colon V \times \mathbb{R}^{n-2} \to \mathbb{R}^n$ as follows:

$$\Phi(u, v, t_3, \ldots, t_n) := (x_1(u, v), x_2(u, v), x_3(u, v) + t_3, \ldots, x_n(u, v) + t_n)$$

Clearly, Φ is smooth. The Jacobian of Φ at $(q, 0)$ is

$$D\Phi(q, t) = \begin{pmatrix} \frac{\partial x_1}{\partial u} & \frac{\partial x_1}{\partial v} & 0 & \ldots & 0 \\ \frac{\partial x_2}{\partial u} & \frac{\partial x_2}{\partial v} & 0 & \ldots & 0 \\ \frac{\partial x_3}{\partial u} & \frac{\partial x_3}{\partial v} & 1 & \ldots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \frac{\partial x_n}{\partial u} & \frac{\partial x_n}{\partial v} & 0 & \ldots & 1 \end{pmatrix} (q)$$

Clearly this is nonsingular at $(q, 0)$. Hence by the inverse function theorem there exists a neighbourhood $V' := V_1 \times V_2 \ni (q, 0)$ in $\mathbb{R}^n$ such that Φ is 1-1 on V', the image $U' := \Phi(V')$ is open in $\mathbb{R}^n$ containing $\Phi(q, 0) = p$ and $\Phi^{-1}\colon U' \to V'$ is smooth. Observe that for points $(q', 0) \in V'$, $\Phi(q', 0) = \varphi(q') \in U' \cap U$. Hence if $x(u) \in U \cap U'$, then $\Phi^{-1}(x(u)) = \varphi^{-1}(x(u)) = (u, v)$. Since $F\colon W \to U$ is continuous there exists a open set $W' \ni w$ such that $F(W') \subset U \cap U'$. As $F(W') \subset U \cap U'$, we have $p' = F(w') \in U \cap U'$ and so $\Phi^{-1}(p') = \varphi^{-1}(p')$ and hence $\varphi^{-1} \circ F = \Phi^{-1} \circ F$ on W'. Since F and Φ^{-1} are smooth, $\Phi^{-1} \circ F$ and hence $\varphi^{-1} \circ F$ are smooth on W'. Since $w \in W$ was arbitrary, the theorem is proved. □

Corollary 3. *Let $S \subset \mathbb{R}^n$ be a surface. Let (V_i, φ_i, U_i), $i = 1, 2$ be two patches of S. Then the map $\varphi_2^{-1} \circ \varphi_1$ as a map from the open set $\varphi_1^{-1}(U_1 \cap U_2)$ in $\mathbb{R}^2$ to $\mathbb{R}^2$ is smooth.* □

Theorem 4. *Let $S \subset \mathbb{R}^3$ be a surface. Then it is locally a graph of a function defined on an open subset $V \subset \mathbb{R}^2$.*

Proof. What we are asked to prove is that given a point $p \in S$, we need to establish the existence of an open set $V' \subset \mathbb{R}^2$, a smooth function $f\colon V' \to \mathbb{R}$ and an open set $U' \ni p$ in S such that U' is the graph of f.

We choose a patch at p, say, (V, φ, U). As usual, we write $\varphi(u,v) = (x(u,v), y(u,v), z(u,v))$. Let $q \in V$ be such that $\varphi(q) = p$. Since the rank of $D\varphi(q')$ is 2 at all $q' \in V$, we assume without loss of generality that the submatrix

$$\begin{pmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{pmatrix}(q)$$

has nonzero determinant. Consider the map $\psi\colon V \to \mathbb{R}^2$ given by $\psi(u,v) = (x(u,v), y(u,v))$. This map is smooth and has a nonsingular Jacobian at $q \in V$. Hence by the inverse function theorem, there exists an open set V' in V containing q with the following properties: (i) ψ is 1-1 on V', (ii) the image $\psi(V')$ is an open set, say W, containing (p_1, p_2) where $p = (p_1, p_2, p_3)$ and (iii) the inverse $\psi^{-1}\colon W \to V'$ is smooth. Let (s,t) be the coordinates in W. Note that $(s,t) = (x(u,v), y(u,v))$ for a unique $(u,v) \in V'$. The map

$$(s,t) \mapsto \psi^{-1}(s,t) = (u,v) \mapsto (x(u,v), y(u,v), z(u,v)) \mapsto z(u,v)$$

is smooth, as its is the composition $\pi \circ \varphi \circ \psi^{-1}$. Call it f. It is clear that the graph $\{(s,t,f(s,t)) : (s,t) \in W\} = \varphi(V')$. Since φ is a homeomorphism, $\varphi(V')$ is open in S containing p. □

Theorem 5. *Let $S \subset \mathbb{R}^n$ be a surface. Let $f\colon S \to \mathbb{R}$ be smooth. Given $p \in S$, there exists an open set $U \ni p$ in $\mathbb{R}^n$ and a smooth function $g\colon U \to \mathbb{R}$ such that $f = g|_S$.*

Proof. We keep the notation in the proof of Theorem 2. Proceeding as in the proof over there, we arrive V', Φ, U'. Now consider the function $g\colon U' \to \mathbb{R}$ defined by

$$x = (x_1, \dots, x_n) \mapsto (u, v, t_3, \dots, t_n) := \Phi^{-1}(x) \mapsto (u,v) \mapsto \varphi(u,v) \mapsto f \circ \varphi(u,v).$$

That is, $g = (f \circ \varphi) \circ \pi \circ \Phi^{-1}$, where $\pi\colon \mathbb{R}^2 \times \mathbb{R}^{n-2} \to \mathbb{R}^2$ is the projection. Since f is assumed to be smooth on S, and φ is a patch, the composite $f \circ \varphi$ is smooth. It follows that g is smooth on U'. Also, $g(p') = f(p')$ for $p' \in U' \cap S$. $t_j = 0$ for $j \geq 3$. □