Limit Inferior and Limit Superior

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1. **Limit Inferior and Limit Superior.** Given a bounded sequence (a_n) of real numbers, let $A_n := \{x_k : k \geq n\}$. Consider the numbers

$$s_n := \inf\{a_k : k \geq n\} \equiv \inf A_n \text{ and } t_n := \sup\{a_k : k \geq n\} \equiv \sup A_n.$$

If $|x_k| \leq M$ for all n, then $-M \leq s_n \leq t_n \leq M$ for all n. The sequence (s_n) is an increasing sequence of reals bounded above while (t_n) is a decreasing sequence of reals bounded below. Let

$$\liminf a_n := \lim s_n \equiv \text{l.u.b. } \{s_n\} \text{ and } \limsup a_n := \lim t_n \equiv \text{g.l.b. } \{t_n\}.$$

In case, the sequence (a_n) is not bounded above, then its $\limsup$ is defined to be $+\infty$. Similarly, the $\liminf$ of a sequence not bounded below is defined to be $-\infty$.

2. Let (x_n) be the sequence where $x_n = (-1)^{n+1}$. Then $\liminf x_n = -1$ and $\limsup x_n = 1$.

3. For any bounded sequence (x_n), we have $\liminf x_n \leq \limsup x_n$. *Hint:* $s_n \leq t_n$.

4. Let (a_n) be a bounded sequence of real numbers with $t := \limsup a_n$. Let $\varepsilon > 0$. Then
   (a) There exists $N \in \mathbb{N}$ such that $a_n < t + \varepsilon$ for $n \geq N$.
   (b) $t - \varepsilon < a_n$ for infinitely many n.
   (c) In particular, there exists infinitely many $r \in \mathbb{N}$ such that $t - \varepsilon < a_r < t + \varepsilon$.

   Proof. Let $A_k := \{x_n : n \geq k\}$.

   (a) Note that $\limsup a_n = \inf t_n$ in the notation used above. Since $t + \varepsilon$ is greater than the greatest lower bound of (t_n), $t + \varepsilon$ is not a lower bound for t_n's. Hence there exists $N \in \mathbb{N}$ such that $t + \varepsilon > t_N$. Since t_N is the least upper bound for $\{x_n : n \geq N\}$, it follows that $t + \varepsilon > x_n$ for all $n \geq N$.

   (b) $t - \varepsilon$ is less than the greatest lower bound of t_n's and hence is certainly a lower bound for t_n's. Hence, for any $k \in \mathbb{N}$, $t - \varepsilon$ is less than t_k, the least upper bound of $\{a_n : n \geq k\}$. Therefore, $t - \varepsilon$ is not an upper bound for $\{a_n : n \geq k\}$. Thus, there exists n_k such that $a_{n_k} > t - \varepsilon$. For $k = 1$, let n_1 be such that $a_{n_1} > t - \varepsilon$. Since $t - \varepsilon$ is not an upper bound of A_{n_1+1} there exists $n_2 \geq n_1 + 1 > n_1$ such that $t - \varepsilon < a_{n_2}$. Proceeding this way, we get a subsequence (a_{n_k}) such that $t - \varepsilon < a_{n_k}$ for all $k \in \mathbb{N}$. □

5. Analogous results for $\liminf$: Let (a_n) be a bounded sequence of real numbers with $s := \liminf a_n$. Let $\varepsilon > 0$. Then
   (a) There exists $N \in \mathbb{N}$ such that $a_n > t - \varepsilon$ for $n \geq N$.
   (b) $t + \varepsilon > a_n$ for infinitely many n.
   (c) In particular, there exists infinitely many $r \in \mathbb{N}$ such that $s - \varepsilon < a_r < s + \varepsilon$.

6. Understand the last two results by applying them to the sequence with $x_n = (-1)^{n+1}$.

7. A sequence (x_n) in $\mathbb{R}$ is convergent iff (i) its bounded and (ii) $\limsup x_n = \liminf x_n$, in which case $\lim x_n = \limsup x_n = \liminf x_n$.

   Proof. Assume that $x_n \to x$. Then (x_n) is bounded. Then $s = \liminf x_n$ and $t = \limsup x_n$ exist. We need to show that $s = t$. Note that $s \le t$. Let $\varepsilon > 0$ be given. Then there exists $N \in \mathbb{N}$ such that

$$n \ge N \implies x - \varepsilon < x_n < x + \varepsilon.$$

   In particular, $x - \varepsilon < s_N := \inf\{x_n : n \ge N\}$ and $t_N := \sup\{x_n : n \ge N\} < x + \varepsilon$. But we have

$$s_N \le \liminf x_n \le \limsup x_n \le t_N.$$

   Hence it follows that

$$x - \varepsilon < s_N \le s \le t \le t_N < x + \varepsilon.$$

   Thus, $|s-t| \le 2\varepsilon$. This being true for all $\varepsilon > 0$, we deduce that $s = t$. Also, $x, s \in (x-\varepsilon, x+\varepsilon)$ for each $\varepsilon > 0$. Hence $x = s$ and hence $x = s = t$.

   Let $s = t$ and $\varepsilon > 0$ be given. Using Items 5 and 4, we see that there exists $N \in \mathbb{N}$ such that

$$n \ge N \implies s - \varepsilon < x_n \text{ and } x_n < s + \varepsilon.$$

   □

8. A traditional proof of the Cauchy completeness of $\mathbb{R}$ runs as follows.

   Proof. Let (x_n) be a Cauchy sequence of real numbers. Then it is bounded and hence $s = \liminf x_n$ and $t = \limsup x_n$ exist as real numbers. It suffices to show that $s = t$. Since $s \le t$ always, we need only show that $t \le s$, that is, $t \le s + \varepsilon$ for any give $\varepsilon > 0$. Since (x_n) is Cauchy there exists $N \in \mathbb{N}$ such that

$$m, n \ge N \implies |x_n - x_m| < \varepsilon/2, \text{ in particular, } |x_n - x_N| < \varepsilon/2.$$

   It follows that for $n \ge N$,

$$x_N - \varepsilon/2 \le \text{g.l.b. } \{x_n : n \ge N\} \le \text{l.u.b. } \{x_n : n \ge N\} \le x_N + \varepsilon/2.$$

   Hence, we obtain

$$t_n := \text{l.u.b. } \{x_n : n \ge N\} \le \text{g.l.b. } \{x_n : n \ge N\} + \varepsilon = s_n + \varepsilon, \text{ for } n \ge N.$$

   Taking limits, we get $\lim t_n \le \lim s_n + \varepsilon$. □

9. Exercises on limit superior and inferior.

   (a) Consider $(x_n) := (1/2, 2/3, 1/3, 3/4, 1/4, 4/5, \ldots, 1/n, n/(n+1), \ldots)$. Then $\limsup = 1$ and $\liminf x_n = 0$.

   (b) Find the limsup and liminf of the sequences whose n-th term is given by:

      i. $x_n = (-1)^n + 1/n$

ii. $x_n = 1/n + (-1)^n/n^2$
iii. $x_n = (1 + 1/n)^n$
iv. $x_n = \sin(n\pi/2)$.

There is a formula for the radius of convergence of a power series $\sum_{n=0}^{\infty} a_n(z-a)^n$ in terms of the coefficients a_n.

10. ***Hadamard formula for the radius of convergence** The radius of convergence ρ of $\sum_{n=0}^{\infty} c_n(z-a)^n$ is given by

$$\frac{1}{\rho} = \limsup |c_n|^{1/n} \text{ and } \rho = \liminf |c_n|^{-1/n}.$$

Proof. Let $\frac{1}{\beta} := \limsup |c_n|^{1/n}$. We wish to show that $\rho = \beta$.

If z is given such that $|z-a| < \beta$, choose μ such that $|z-a| < \mu < \beta$. Then $\frac{1}{\mu} > \frac{1}{\beta}$ and hence there exists N (by the last lemma) such that $|c_n|^{1/n} < \frac{1}{\mu}$ for all $n \geq N$. It follows that $|c_n|\mu^n < 1$ for $n \geq N$. Hence $(|c_n|\mu^n)$ is bounded, say, by M. Hence, $|c_n| \leq M\mu^{-n}$ for all n. Consequently,

$$|c_n(z-a)^n| \leq M\mu^{-n}|z-a|^n = M\left(\frac{|z-a|}{\mu}\right)^n.$$

Since $\frac{|z-a|}{\mu} < 1$, the convergence of $\sum c_n(z-a)^n$ follows.

Let $|z-a| > \beta$ so that $\frac{1}{|z-a|} < \frac{1}{\beta}$. Then $\frac{1}{|z-a|} < |c_n|^{1/n}$ for infinitely many n. Hence $|c_n||z-a|^n \geq 1$ for infinitely many n so that the series $\sum c_n(z-a)^n$ is divergent. We therefore conclude that $\rho = \beta$.

The other formula for the radius of convergence is proved similarly. □