# Linear ODE: Systems of First Order and $n$-th Order

S. Kumaresan
School of Math. and Stat.
University of Hyderabad
Hyderabad 500046
kumaresa@gmail.com

## 1 A Fundamental Existence Theorem for Matrix Equations

**Theorem 1.** *Let $J := [t_0, t_1] \subset \mathbb{R}$ be an interval. Let $A, F \colon J \to M(n, \mathbb{R})$ be continuous $n \times n$ matrix valued functions. Then the matrix DE*

$$X'(t) = A(t)X(t) + F(t), \qquad X(t_0) = X_0, \tag{1}$$

*for a given matrix $X_0$ has a unique solution on the entire interval $J$.*

*Proof.* The given problem is equivalent to solving the matrix valued integral equation

$$X(t) = X_0 + \int_{t_0}^{t} (A(s)X(s) + F(s))\, ds. \tag{2}$$

We adopt the Picard iteration scheme:

$$X_0(s) := X_0, \quad X_m(t) := X_0 + \int_{t_0}^{t} (A(s)X_{m-1}(s) + F(s))\, ds.$$

Let $M > 0$ be such that $\| A(s) \| \leq M$ for $s \in J$. We prove by induction that

$$\| X_m(t) - X_{m-1}(t) \| \leq \frac{M^m}{m!}(t - t_0)^m, \tag{3}$$

for all $t \in J$ and $m \in \mathbb{N}$. As in the case of Picard's local existence theorem, we conclude that $(X_m)$ is uniformly Cauchy on $J$ and hence they converge to an $X$ which satisfies the integral equation (2).

We now prove that the solution is unique. Let $X_1$ and $X_2$ be solutions of the DE (1). Then their difference $Y := X_1 - X_2$ is a solution of the homogeneous problem IV problem

$$Y'(t) = A(t)Y(t), \qquad Y(t_0) = 0. \tag{4}$$

If we set $g(t) := \int_{t_0}^t \|Y(s)\|\,ds$, we see that

$$\begin{aligned} g'(t) = \|Y(t)\| = \|Y(t) - Y(t_0)\| &= \left\| \int_{t_0}^t Y'(s) \right\| \\ &\le \int_{t_0}^t \|Y'(s)\|\,ds \\ &= \int_{t_0}^t \|A(s)Y(s)\|\,ds \\ &\le M \int_{t_0}^t \|Y(s)\| \\ &= Mg(t). \end{aligned}$$

Hence we have

$$g'(t) - Mg(t) \le 0, \qquad (t \in J).$$

But then

$$\frac{d}{dt}(g(t)e^{-Mt}) = [g'(t) - Mg(t)]e^{-Mt} \le 0, \qquad (t \in J),$$

so that the function $t \mapsto g(t)e^{-Mt}$ is decreasing. Since $g(t_0) = 0$ and is nonnegative by definition, it follows that $g = 0$ on $J$. □

A matrix DE $X'(t) = A(t)X(t) + F(t)$ is said to be homogeneous if $F = 0$.

**Theorem 2.** *Let the notation be as in the last theorem. Let $X$ be a solution of (1). The* **Wronskian** $W(t) := \det X(t)$ *of the solution satisfies the scalar DE*

$$W'(t) = (\operatorname{Tr} A(t))W(t).$$

*This implies that if $X(\tau)$ is invertible for some $\tau \in J$, then $X(t)$ is invertible for all $t \in J$.*

*Proof.* We use Ex. 3 given below. Let $x_1(t), \dots, x_n(t)$ be the columns of $X(t)$. We have

$$\begin{aligned} W'(t) &= \sum_{k=1}^n \det(x_1, \dots, x_k'(t), \dots, x_n(t)) \\ &= \sum_{k=1}^n \det(x_1, \dots, A(t)x_k(t), \dots, x_n(t)) \\ &= \operatorname{Tr}(A(t)) \det(x_1(t), \dots, x_n(t)) \\ &= \operatorname{Tr}(A(t))W(t). \end{aligned}$$

Integrating this DE, we find that $W(t) = ce^{\int_{t_0}^t \operatorname{Tr} A(s)\,ds}$ for the constant $c = W(t_0)$. The last statement of the theorem follows from this observation. □

**Ex. 3.** Let $x_i : J \to \mathbb{R}^n$ be differentiable functions. Let $X(t) := (x_1(t), \dots, x_n(t))$ where $x_j$ is considered as the $j$-th column of $X$. Show that

$$\frac{d}{dt}\det(X(t)) = \sum_{k=1}^n \det(x_1(t), \dots, x_k'(t), \dots, x_n(t)).$$

*Hint:* Recall that det is a multilinear function. The derivative can be found from the first principles. Or, use the Laplace expansion.

## 2 Linear Systems: $x' = Ax + f$

We use the above theorem to deduce all the results concerning the first order system

$$x'(t) = A(t)x(t) + f(t),$$

where $A\colon J \to M(n, \mathbb{R})$ and $f\colon J \to \mathbb{R}^n$ are continuous functions. Written componentwise, we have

$$x_i'(t) = a_{i1}(t)x_1(t) + \cdots + a_{in}(t)x_n(t) + f_i(t), \quad (1 \le i \le n).$$

**Theorem 4.** *Let $J := [t_0, t_1] \subset \mathbb{R}$ be an interval. let $A\colon J \to M(n, \mathbb{R})$ and $f\colon J \to \mathbb{R}^n$ be continuous functions. Consider the nonhomogeneous first order system*

$$x'(t) = A(t)x(t) + f(t) \tag{5}$$

*and the corresponding homogeneous system*

$$x'(t) = A(t)x(t) \tag{6}$$

(a) *For any given vector $x_0 \in \mathbb{R}^n$, the system (5) has a unique solution with $x(t_0) = x_0$.*
(b) *If $x_1(t), \dots, x_n(t)$ are $n$ solutions of (6), then the following are equivalent.*
(i) *$x_1., \dots, x_n$ are linearly dependent on $J$.*
(ii) *$\det(x_1(t), \dots, x_n(t)) = 0$ for all $t \in J$.*
(iii) *$\det(x_1(t), \dots, x_n(t)) = 0$ for some $t \in J$.*
(c) *The solution set of the homogeneous system $x'(t) = A(t)x(t)$ is a vector space of dimension $n$. Any basis of the space of solutions is known as a fundamental set.*
(d) *If $x_p$ is a particular solution of the nonhomogeneous system (5), then any solution is of the form $x_p + x$ where $x$ is a solution of the associated homogeneous system.*
(e) *Let $(x_1, \dots, x_n)$ be a fundamental set of the homogeneous system. Let $X(t) := (x_1(t), \dots, x_n(t))$ be the matrix whose $i$-th column is $x_i(t)$. If $c(t)$ is any solution of the equation*

$$X(t)c'(t) = f(t)$$

*then*

$$x(t) := X(t)c(t)$$

*is a solution of the nonhomogeneous equation.*
(f) *Let $(x_1, \dots, x_n)$ be a fundamental set for the homogeneous system. Assume that $u_k$ are real valued functions such that*

$$u_k'(t) = \frac{\det(x_1(t), \dots, x_{k-1}(t), f(t), x_{k+1}(t), \dots, x_n(t))}{\det(x_1(t), \dots, x_n(t))}.$$

*Then $x(t) := u_1(t)x_1(t) + \cdots + u_n(t)x_n(t)$ is a solution of the nonhomogeneous system (5).*

*Proof.* (a) follows trivially, if we apply Theorem 1 to $X_0 := (x_0, \dots, x_0)$ and $F(t) := (f(t), \dots, f(t))$. Then $X(t)$ is of the form $X(t) = (x(t), \dots, x(t))$.

(b) Let $X(t) := (x_1(t), \dots, x_n(t))$. Then $x_1, \dots, x_n$ are solutions of $x' = Ax$ iff $X$ is a solution of $X' = AX$.

(c) Let $x_k$ be the unique solution of $x' = Ax$ with $x(t_0) = e_k$, the $k$-th basic vector of $\mathbb{R}^n$. (Existence of $x_k$ is assured by (a). ) Let $y$ be any solution of the homogeneous system. Then $y(t_0) = \sum_k c_k e_k$. If we let $x(t) := \sum_k c_k x_k$, then $x$ is a solution of the homogeneous system such that $x(t_0) = y(t_0)$. By the uniqueness, it follows that $y = x$.

(d) is trivial, if we observe that the difference of any two solutions of the nonhomogeneous system is a solution of the homogeneous system.

(e) If $x(t) = X(t)c(t)$, then by the product rule, we have

$$x'(t) = X'(t)c(t) + X(t)c'(t) = A(t)X(t)c'(t) + f(t) = A(t)x(t) + f(t).$$

(f) For any fixed $t \in J$, the vector $c(t) := (c_1(t), \ldots, c_n(t))$ (considered as a column vector) is, by Cramer's rule, the solution of the linear system $X(t)c'(t) = f(t)$. Hence $t \mapsto X(t)c(t)$ is a solution of the nonhomogeneous equation (1) by (e). But,

$$X(t)c(t) = c_1(t)x_1(t) + \cdots + c_n(t)x_n(t),$$

by the definition of matrix multiplication. □

**Remark 5.** Note that the equation $X(t)c'(t) = f(t)$ in (e) of the above theorem says that $c$ is an anti-derivative of $X(t)^{-1}f(t)$, e.g., $c(t) = \int_{t_0}^{t_1} X(s)^{-1}f(s)\,ds$. Thus (f) of the last theorem allows us to solve for the nonhomogeneous system given a fundamental set for the homogeneous system. This is known as the *method of variation of parameters.*

If the matrix function $A(t) = A$ is a constant matrix, we then have an explicit representation of the solution of the non-homogeneous equation.

**Theorem 6.** *Let $A$ be a constant matrix. Let $J = [t_0, t_1] \subset \mathbb{R}$ be an interval. Let $f \colon J \to \mathbb{R}^n$ be continuous. Then the unique solution of the initial value problem*

$$x'(t) = Ax(t) + f(t), \qquad x(t_0) = x_0$$

*is given by*

$$x(t) = e^{(t-t_0)A}x_0 + \int_{t_0}^{t} e^{(t-s)A}f(s)\,ds.$$

*Proof.* Multiply the DE by $e^{-tA}$ on the left to obtain

$$e^{-tA}x'(t) - e^{-tA}Ax(t) = e^{-tA}f(t).$$

The LHS of this equation is the derivative the function $t \mapsto e^{-tA}x(t)$. So, upon integration, we get

$$e^{-tA}x(t) = \int_{t_0}^{t_1} e^{-sA}f(s)\,ds + c.$$

The constant vector is identified by taking $t$ as $t_0$. □

Let $A$ be a constant matrix. We want to solve $x' = Ax$ with IC $x(0) = x_0$ explicitly. This is easily done. Let $x(t) := e^{tA}x_0$ where $e^{tA} := \sum_{k=0}^{\infty} \frac{(tA)^k}{k!}$. Then the standard results about the exponential of matrices tell us that the problem is solved. (See Exer. 7 below.) However, it is usually very difficult to compute the exponential of any matrix $A$. Before attending to this, we establish some useful results.

**Ex. 7** (Exponential Map in $M(n,\mathbb{R})$)**.** The following set of exercises introduces the exponential map in $M(n,\mathbb{R})$ and its properties:

1. For $X \in M(n,\mathbb{R})$, $X := (x_{ij})$, let
$$\| X \|_\infty := \max_{1\le i,j\le n} |x_{ij}|$$
be the max norm. It is equivalent to the operator norm $\| \ \|$ on elements of $M(n,\mathbb{R})$ *viewed as linear operators on* $\mathbb{R}^n$. We shall use the operator norm in the following.

2. We have $\| AB \| \le \| A \| \, \| B \|$ for all $A, B \in M(n,\mathbb{R})$ and $\left\| A^k \right\| \le \| A \|^k$.

3. A sequence $A_k \to A$ in the operator norm if and only if $a_{ij}^k \to a_{ij}$ for all $1 \le i, j \le n$ as $k \to \infty$. Here we have $A_k := (a_{ij}^k)$, etc.

4. If $\sum_{k=0}^\infty \| A_k \|$ is convergent, then $\sum_{k=0}^\infty A_k$ is convergent to an element $A$ of $M(n,\mathbb{R})$.

5. For any $X \in M(n,\mathbb{R})$, the series $\sum_{k=0}^\infty \frac{X^k}{k!}$ is convergent. We denote the sum by $\exp(X)$ or by $e^X$.

6. For a fixed $X \in M(n,\mathbb{R})$ the function $f(t) := e^{tX}$ satisfies the matrix differential equation $f'(t) = Xf(t)$, with the initial value $f(0) = I$. *Hint:* Note that the $(i,j)$-th entry of $f(t)$ is a power series in $t$ and use (4).

7. Set $g(t) := e^{tX} e^{-tX}$ and conclude that $e^{tX}$ is invertible for all $t \in \mathbb{R}$ and for all $X \in M(n,\mathbb{R})$.

8. There exists a unique solution for $f'(t) = Af(t)$ with initial value $f(0) = B$ given by $f(t) = e^{tA}B$. *Hint:* If $g$ is any solution, consider $h(t) = g(t)\, e^{-tA}$.

9. Let $A$, $B \in M(n,\mathbb{R})$. If $AB = BA$ then we have
$$e^{A+B} = e^A e^B = e^B e^A = e^{B+A}.$$
*Hint:* Consider $\phi(t) := e^{t(A+B)} - e^{tA} e^{tB}$.

10. For $A$, $X \in M(n,\mathbb{R})$ we have $e^{AXA^{-1}} = Ae^X A^{-1}$.

**Definition 8.** We say that a matrix $X(t)$ with columns $x_i(t)$ is a *fundamental matrix* if $\{x_i : 1 \le i \le n\}$ is a basis of solutions of the DE $x' = Ax$. Note that by the fundamental theorem for linear systems such a matrix exists.

**Lemma 9.** *A matrix $X(t)$ is a fundamental matrix for $x' = Ax$ iff $X'(t) = AX(t)$ and* $\det X(t) \neq 0$.

*Proof.* Let $x_j$ be the $j$-th column of $X$. Observe that the matrix equation $X' = AX$ is equivalent the $n$ vector equations $x_j'(t) = Ax_j(t)$. By the standard uniqueness argument, the $n$ solutions $x_j(t)$ are linearly independent iff $x_1(0), \ldots, x_n(0)$ are linearly independent. The latter are linearly independent iff $\det X(0) \neq 0$. □

**Lemma 10.** *The matrix $e^{tA}$ is a fundamental solution of $x' = Ax$.*

*Proof.* Obvious in view of the last lemma. □

**Lemma 11.** *Let $X(t)$ and $Y(t)$ be two fundamental solutions of $x' = Ax$. Then there exists a constant matrix $C$ such that $Y(t) = X(t)C$.*

*Proof.* Each column $y_i$ of $Y$ can be written as a linear combination of the columns $x_j$ of $X$:

$$y_i = c_{1i}x_1 + \cdots + c_{ni}x_n.$$

Then $C := (c_{ij})$ is as required. □

**Theorem 12.** *Let $X(t)$ be a fundamental matrix of $x' = Ax$. Then*

$$e^{tA} = X(t)X^{-1}(0). \tag{7}$$

*In other words, any fundamental matrix $X(t)$ is of the form $X(t) = e^{tA}X(0)$.*

*Proof.* Immediate consequence of the last two lemmas. □

**Ex. 13.** Let $x_j$ be the solution of the initial value problem $x' = Ax$ with $x_j(0) = e_j$. Show that $e^{tA} = (x_1, \ldots, x_n)$.

**Ex. 14.** Let $X$ and $Y$ be fundamental matrices of $x' = Ax$ with $Y = XC$ for a constant matrix $C$. Show that $\det C \neq 0$.

**Ex. 15.** Let $X(t)$ be a fundamental matrix of $x' = Ax$ and $C$ a constant matrix with $\det C \neq 0$. Show that $Y(t) = X(t)C$ is a fundamental matrix of $x' = Ax$.

**Ex. 16.** Let $X$ be a fundamental solution of $x' = Ax$. Prove that the solution of the IV problem $x' = Ax$, $x(t_0) = x_0$ is $x(t) = X(t)(X(t_0)^{-1}x_0$.

**Ex. 17.** Let $X$ be a fundamental matrix of $x' = Ax$. Show that $X(t)X(t_0)^{-1} = e^{(t-t_0)A}$.

**Theorem 18** (Nonhomogeneous Equation-Variation of Parameters)**.** *The solution of the IV problem $x' = Ax + f(t)$, $x(t_0) = x_0$ is given by*

$$x(t) = X(t)X^{-1}(t)x_0 + X(t)\int_{t_0}^{t} X^{-1}(s)f(s)\,ds, \tag{8}$$

*where $X$ is a any fundamental matrix of the homogeneous equation $x' = Ax$.*

*Proof.* Let $x_1, \ldots, x_n$ be a set of $n$ linearly independent solutions of the homogeneous system $x' = Ax$. We seek a solution $x$ of the IV problem for the nonhomogeneous system in the form

$$x(t) = u_1(t)x_1(t) + \cdots + u_n(t)x_n(t).$$

This can be written as $x(t) = X(t)u(t)$ in an obvious notation. Assuming that $x(t)$ solves the IV problem and plucking the expression for $x(t)$ in the equation $x' = Ax + f$, we get

$$X'(t)u(t) + X(t)u'(t) = AX(t)u(t) + f(t). \tag{9}$$

Since $X$ is a fundamental matrix $X'(t) = AX(t)$ so that the first terms on either side of (9) are equal. Hence, (9) reduces to

$$X(t)u'(t) = f(t).$$

Thus, $u' = X^{-1}(t)f(t)$ so that

$$\begin{aligned} u(t) &= u(t_0) + \int_{t_0}^{t} X^{-1}(s)f(s)\,ds \\ &= X^{-1}(t_0)(x_0) + \int_{t_0}^{t} X^{-1}(s)f(s)\,ds. \end{aligned}$$

The result (8) follows from this. □

**Remark 19.** Theorem 6 is a special case of the last theorem if we take $X(t) = e^{tA}$. Note that the Green's kernel in this case is $G(t,s) = e^{(t-s)A}$.

# 3 Linear Equations of Higher Order

An $n$-th order linear ODE is of the form

$$y^{(n)} + a_{n-1}y^{(n-1)} + \cdots + a_1 y' + a_0 y = f(t), \tag{10}$$

where the coefficient function $a_j$ and $f$ are assumed to be continuous functions on an interval $J \subset \mathbb{R}$. The homogeneous linear equation associated to (10) is

$$y^{(n)} + a_{n-1}y^{(n-1)} + \cdots + a_1 y' + a_0 y = 0. \tag{11}$$

The crucial observation is that the study of such equations can be reduced to the study of first order systems considered above. If $y$ is a solution of the DE (10), then the functions

$$x_1 := y, x_2 := y', x_3 := y'', \ldots, x_{n-1} = y^{(n-2)}, x_n := y^{(n-1)}$$

satisfy the following differential equations

$$x_1' = x_2, x_2' = x_3, \ldots, x_{n-1}' = x_n, x_n' = -(a_{n-1}x_n + a_{n-2}x_{n-1} + \cdots + a_1 x_2 + a_0 x_1) + f. \tag{12}$$

We introduce the matrix valued function

$$A(t) := \begin{pmatrix} 0 & 1 & 0 & \ldots & 0 \\ 0 & 0 & 1 & \ldots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \ldots & 1 \\ -a_0 & -a_1 & \ldots & \ldots & -a_{n-1} \end{pmatrix}.$$

With this notation, the differential equations in (12), can be recast as

$$\begin{pmatrix} x_1'(t) \\ x_2'(t) \\ \vdots \\ x_{n-1}'(t) \\ x_n'(t) \end{pmatrix} = A(t) \begin{pmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_{n-1}(t) \\ x_n(t) \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ f(t) \end{pmatrix}$$

or, in an obvious notation

$$x'(t) = A(t)x(t) + F(t). \tag{13}$$

**Ex. 20.** Every solution $y$ of (10) is a solution of (13). Conversely, if $x(t)$ is a solution of (13), then $y(t) := x_1(t)$ is a solution of (10).

The matrix $A$ above is called the *companion matrix* of DE (10).

The following theorem is more or less an immediate consequence of Theorem 4.

**Theorem 21.** *Let $J := [t_0, t_1] \subset \mathbb{R}$ be an interval. Let $a_0, a_1, \ldots, a_{n-1} \colon J \to \mathbb{R}$ and $f \colon J \to \mathbb{R}$ be continuous. Let $L(y) := y^{(n)}(t) + a_{n-1}y^{(n-1)}(t) + \cdots + a_1(t)y' + a_0(t)y(t)$. Consider the nonhomogeneous equation $L(y) = f$ and the homogeneous equation $L(y) = 0$.*

(a) *Given any numbers $\beta_j \in \mathbb{R}$, for $0 \le j \le k-1$, there is a unique solution $y$ of $L(y) = f$ with $y^{(j)}(t_0) = \beta_j$.*

(b) *If $y_1, \ldots, y_n$ are solutions of the homogeneous equation $L(y) = 0$, then their Wronskian determinant*

$$W(y_1, \ldots, y_n)(t) := \begin{pmatrix} y_1(t) & y_2(t) & \ldots & y_n(t) \\ y_1'(t) & y_2'(t) & \ldots & y_n'(t) \\ \vdots & \vdots & & \vdots \\ y_1^{(n-1)}(t) & y_2^{(n-1)}(t) & \ldots & y_n^{(n-1)}(t). \end{pmatrix}$$

*satisfies the differential equation: $W'(t) = -a_{n-1}(t)W(t)$.*

(c) *If $y_1, \ldots, y_n$ are $n$ solutions of the homogeneous equation $L(y) = 0$, then the following are equivalent:*

*(i) $y_1, \ldots, y_n$ are linearly independent on $J$.*

*(ii) $W(y)(t) = 0$ for all $t \in J$.*

*(iii) $W(y)(\tau) = 0$ for some $\tau \in J$.*

(d) *The set of solutions of the homogeneous equation $L(y) = 0$ is an $n$-dimensional vector space. Any basis of the space of solutions is called a fundamental set.*

(e) *If $y_p$ is a particular solution of the nonhomogeneous system, then any solution of the nonhomogeneous system is of the form $y_p + y$ where $y$ is a solution of the homogeneous system.*

(f) *Let $\{y_j : 1 \le j \le n\}$ be a fundamental set of the space of solutions of $L(y) = 0$. If $u_1, \ldots, u_n$ are solutions of the matrix equation*

$$\begin{pmatrix} y_1 & y_2 & \ldots & y_n \\ y_1' & y_2' & \ldots & y_n' \\ \vdots & \vdots & & \vdots \\ y_1^{(n-1)} & y_2^{(n-1)} & \ldots & y_n^{(n-1)}. \end{pmatrix} \begin{pmatrix} u_1'(t) \\ u_2'(t) \\ \vdots \\ u_n'(t) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ f(t) \end{pmatrix}$$

*then*

$$y(t) := u_1(t)y_1(t) + \cdots + u_n(t)y_n(t)$$

*is a solution of the nonhomogeneous equation $L(y) = f$.*

Let $D\colon f \mapsto f'$ denote the differential operator. If $p(X) := X^n + a_{n-1}X^{n-1} + \cdots + a_1 X + a_0$, then we let

$$p(D)(y) := y^{(n)} + a_{n-1}y^{(n-1)} + \cdots + a_1 y' + a_0(t)y.$$

The polynomial $p$ is called the *characteristic polynomial* of the differential operator $p(D)$.

**Theorem 22.** *Given a differential operator*

$$p(D)(y)(t) := y^{(n)}(t) + a_{n-1}y^{(n-1)}(t) + \cdots + a_1 y'(t) + a_0 y(t)$$

*with constant coefficients. Let $\lambda_j$ be the roots of $p$ with multiplicity $m_j$, ($1 \le j \le k$). Any solution $y$ of the homogeneous equation $p(D)y = 0$ is of the form*

$$y(t) = e^{\lambda_1 t} p_1(t) + \cdots + e^{\lambda_k t} p_k(t)$$

*where $p_j$ is an arbitrary polynomial of degree at most $m_j$. That is, a basis of the solution is*

$$\{e^{\lambda_j t} t^r : 1 \le j \le k, 0 \le r \le m_j - 1\}.$$

*Proof.* Follows from the next three lemmas. □

**Lemma 23.** *Let $V_r := C^r(\mathbb{R})$ be the vector space of all $r$-times continuously differentiable functions on $\mathbb{R}$. Let $D\colon V_r \to V_{r-1}$ be the derivation map $f \mapsto f'$.*

(a) *If $f \in V_r$, then $(D - \lambda I)^r f = e^{\lambda t} D^r(e^{-\lambda t} f)$.*

(b) *A function $f \in V_r$ lies in the kernel of $(D - \lambda I)^r$ iff it is of the form*

$$f(t) = e^{\lambda t}(b_0 + b_1 t + \cdots + b_{r-1} t^{r-1}).$$

*Proof.* (a) is proved by induction. To prove (b), observe that $(D - \lambda I)^r f = 0$ iff $D^r(e^{-\lambda t} f) = 0$, by (a). □

**Lemma 24.** *Let $p \in \mathbb{K}[X]$ be a polynomial over a field $\mathbb{K}$ with a decomposition $p = p-1 \cdots p_k$ where $p_j$ are relatively prime. Let $A\colon V \to V$ be a linear endomorphism of the $\mathbb{K}$-vector space $V$. Then we have*

$$\ker p(A) = \ker p_1(A) \oplus \cdots \oplus \ker p_k(A).$$

*Proof.* Let $q_j := p/p_j = p_1 \cdots p_{j-1} p_{j+1} \cdots p_k$. We first show that the sum is direct. Let $v_1 + \cdots + v_k = 0$ where $v_j \in \ker p_j(A)$. Note that $q_i(A)v_j = 0$ whenever $i \ne j$. Since $p_i$ and $q_i$ are relatively prime, there exist polynomials $a$ and $b$ such that $ap_1 + bq_1 = 1$. Then,

$$\begin{aligned} v_i &= a(A)p_i(A)v_i + b(A)bq_i(A)v_i \\ &= b(A)q_i(A)(-\sum_{j\ne i} v_j) \\ &= -\sum_{j\ne i} b(A)q_i(A)v_j \\ &= \sum_{j \ne i} 0. \end{aligned}$$

In the displayed equation, the inclusion $\supseteq$ is obvious. To prove the reverse inclusion, note that $q_j$ are relatively prime. Hence there exist polynomials $r_j$ such that $\sum_j r_j q_j = 1$. Let $v \in \ker p(A)$. Then $v_i := r_i(A)q_i(A)v \in \ker p_i(A)$ because

$$p_i(A)v_i = p_i(A)qr_i(A)q_i(A)v = r_i(A)p_i(A)q_i(A) = r_i(A)p(A)v = 0.$$

It is easily seen that $v = \sum_i v_i$. □

**Lemma 25.** *Let $p(\lambda) := \lambda^n + a_{n-1}\lambda^{n-1} + \cdots + a_1\lambda + a_0 = (\lambda - \lambda_1)^{m_1} \cdots (\lambda - gl_k)^{m_k}$. Then a basis solutions of the homogeneous equation $p(D)y = 0$ is $\{e^{\lambda_j t}t^r : 1 \leq j \leq k, 0 \leq r \leq m_j - 1\}$.*

*Proof.* Let $p_i(\lambda) := (\lambda - \lambda_i)^{m_i}$. Then $p(D) = p_1(D) \cdots p_k(D)$. The result follows from the last two lemmas. □

We now apply our knowledge about the $n$-th order homogeneous equation to compute $e^{tA}$ of a matrix $A$!

**Theorem 26.** *Let $p$ be the characteristic polynomial of $A$. Let $y_1, \ldots, y_n$ be a basis for the set of solutions of the homogeneous $n$-th order equation $p(D)y = 0$. Then there exist matrices $A_1, \ldots, A_n$ such that*

$$e^{tA} = y_1(t)A_1 + \cdots + y_n(t)A_n. \tag{14}$$

*Proof.* We have $p(t) := \det(A - tI)$. Let $e^{tA} = (u_{ij}(t))$. Then,

$$(p(D)u_{ij}(t)) = p(D)e^{tA} = p(A)e^{tA} = 0,$$

by Cayley-Hamilton theorem. Thus every entry $u_{ij}$ of $e^{tA}$ satisfies the DE $p(D)u_{ij} = 0$, hence can be written as linear combination of $y_j$'s. □

**Remark 27.** We now give an algorithm to find the exponential of a matrix. Given a matrix $A$, we find its characteristic polynomial $p(t)$. We find a basis $\{y_j : 1 \leq j \leq n\}$ of solutions of $p(D)y = 0$. From the last result we know that there exist matrices $A_j$ such that $e^{tA} = y_1A_1 + \cdots + y_nA_n$. To find these matrices $A_j$, we differentiate the equation (14) $n-1$ times with respect to $t$. We thus obtain $n$ equations a follows.

$$\begin{aligned} e^{tA} &= y_1(t)A_1 + \cdots + y_n(t)A_n \\ Ae^{tA} &= y_1'(t)A_1 + \cdots + y_n'(t)A_n \\ &\vdots \\ A^{n-1}e^{tA} &= y^{(n-1)}(t)A_1 + \cdots + y^{(n-1)}(t)A_n. \end{aligned}$$

Evaluate them at $t = 0$ and solve for $A_j$ by the standard Gaussian elimination method.

**Example 28.** Consider $\begin{pmatrix} 0 & -s \\ s & 0 \end{pmatrix}$. Following the above steps, we obtain $e^{tA} = \begin{pmatrix} \cos st & -\sin st \\ \sin st & \cos st \end{pmatrix}$.

**Ex. 29.** (i) Find $e^{tA}$ where $A = \begin{pmatrix} 0 & 1 \\ -14 & -9 \end{pmatrix}$.

(ii) Transform the equation $y'' + 9y' + 14y = 0$ with $y(0) = 0$ and $y'(0) = -1$ into a vector

DE.

(iii) Solve the IV problem in (ii). Ans: $e^{tA} = \frac{1}{5}\begin{pmatrix} 7e^{-2t} - 2e^{-7t} & e^{-2t} - e^{-7t} \\ -14e^{-2t} + 14e^{-7t} & -2e^{-2t} + 7e^{-7t} \end{pmatrix}$ and $y(t) = \frac{1}{5}(-e^{-2t} + e^{-7t})$.

**Ex. 30.** Solve $x'' + x = 3$ with $x(\pi) = 1$ and $x'(\pi) = 2$ using the methods of exponential matrix and variation of parameters. Ans: $3 + 2\cos t - 2\sin t$.

**Ex. 31.** Solve $x'' + 2x' - 8x = e^t$ with $x(0) = 1$ and $x'(0) = -4$ using the methods of exponential matrix and variation of parameters. Ans: $\frac{31}{30}e^{-4t} + \frac{1}{6}e^{2t} - \frac{1}{5}e^t$.

**Ex. 32.** Compute the exponential of the following matrices by any method:

(i) $\begin{pmatrix} 1 & 1 \\ 4 & 1 \end{pmatrix}$. Ans: $\begin{pmatrix} \frac{1}{2}(e^{-t} + e^{3t}) & \frac{1}{4}(e^{3t} - e^{-t}) \\ -e^{-t} + e^{3t} & \frac{1}{2}(e^{-t} + e^{3t}) \end{pmatrix}$.

(ii) $\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$. Ans: $\begin{pmatrix} e^t \cos t & -e^t \sin t \\ e^t \sin t & e^t \cos t \end{pmatrix}$.

(iii) $\begin{pmatrix} 2 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}$.

(iv) $\begin{pmatrix} 0 & 1 \\ -4 & -4 \end{pmatrix}$. Ans: $e^{-2t}\begin{pmatrix} 1 + 2t & t \\ -4t & 1 - 2t \end{pmatrix}$.

**Ex. 33.** Compute $e^{tA}$ by solving a third order DE where $A = \begin{pmatrix} 2 & 0 & 1 \\ 1 & 0 & 1 \\ 1 & -2 & 0 \end{pmatrix}$.

We now briefly indicate how one can solve a linear system by the eigen value-eigen vector method. Given the system $x' = Ax$ with a constant matrix $A$. Assume that $A$ has distinct eigen values, say, $\lambda_j$, $1 \leq j \leq n$. Let $v_j$ be a nonzero eigen vector of $A$ with eigen value $\lambda_j$. Then $x_j(t) := e^{\lambda_j t} v_j$ is a solution. Also, $\{x_j : 1 \leq j \leq n\}$ is a fundamental set for the equation $x' = Ax$. Even if the eigen values are complex, by Ex. 35, we can find real valued solutions. However, if $A$ has eigen values with multiplicity, one requires a little more work. We refer the reader to Braun's book (especially sections 3.6–3.8) for more details.

**Remark 34.** A fall-out of this approach is another method of computing $e^{tA}$. Let $x_1(t), \ldots, x_n(t)$ be linearly independent solutions of $x' = Ax$. Then the matrix $X(t) := (x_1(t), \ldots, x_n(t))$ is a fundamental matrix so that $e^{tA} = X(t)X^{-1}(0)$. (Ex. 17!)

**Ex. 35.** Let $x(t) = f(t) + ig(t)$ be a solution of $x' = Ax$. Then $f$ and $g$are real valued solutions of $x' = Ax$.

**Ex. 36.** Find all the solutions of $x' = Ax$ using eigen methods where

(i) $A = \begin{pmatrix} 3 & 2 & 4 \\ 2 & 0 & 2 \\ 4 & 2 & 3 \end{pmatrix} x$.

(ii) $A = \begin{pmatrix} 1 & 0 & 0 \\ 3 & 1 & -2 \\ 2 & 2 & 1 \end{pmatrix}$.

**Ex. 37.** Solve the given initial value problem.

(i) $x' = \begin{pmatrix} 1 & 1 \\ 4 & 1 \end{pmatrix} x$, $x(0) = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$.

(ii) $x' = \begin{pmatrix} -3 & 0 & 2 \\ 1 & -1 & 0 \\ -2 & -1 & 0 \end{pmatrix} x$ with $x(0) = \begin{pmatrix} 0 \\ -1 \\ -2 \end{pmatrix}$.

**Ex. 38.** Solve the IV problem

$$x' = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & -2 \\ 3 & 2 & 1 \end{pmatrix} x + \begin{pmatrix} 0 \\ 0 \\ e^t \cos 2t \end{pmatrix}, \qquad x(0) = e_2 + e_3$$

by two different methods viz., by finding the exponential of $A$ and by eigen method.

**Ex. 39.** Find $e^{tA}$ if $A = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 3 & 2 \\ 0 & 0 & 5 \end{pmatrix}$ by methods of Remarks 27 and 34. Which do you find is easier?