Compact Subsets of L^p Spaces

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It is very useful to characterize the compact subsets of concrete metric spaces using intrinsic properties of elements of the spaces. For instance, Heine-Borel theorem gives such a characterization for compact subsets of $\mathbb{R}^n$ and Arzela-Ascoli theorem achieves this for the space $(C(X), \| \ \|_{\sup})$ of continuous functions on a compact space X. Our aim in this article is to give a characterization of compact subsets of $L^p[0,1]$, $1 \le p < \infty$ and later in $L^p(\mathbb{R})$.

Let $f \in L^p[0,1]$. We can consider it as a function on all of $\mathbb{R}$ by setting $f(x) = 0$ for $x \notin [0,1]$.

Ex. 1. Prove that $L^p[0,1] \subset L^1[0,1]$. *Hint:* Apply Hölder's inequality to $\int_0^1 f(x) \cdot 1 \, dx$ where 1 stands for the constant function 1.

For $f \in L^p[0,1]$ and $h > 0$, define

$$f_h(t) := \frac{1}{2h} \int_{t-h}^{t+h} f(x) \, dx.$$

The integral exists thanks to Ex. 1.

Lemma 2. *For $f \in L^p[0,1]$ and $h > 0$, f_h is continuous and $f_h \in L^p[0,1]$.*

Proof. Let $t_n \to t$. Let $g_n := f\chi_{(t_n-h, t_n+h)}$. Then $g_n \to f\chi_{(t-h,t+h)}$ pointwise. Also, we observe that $|g_n| \le |f|$. We apply Lebesgue's dominated convergence theorem to conclude

$$\lim f_h(t_n) = \frac{1}{2h} \lim \int_{t_n-h}^{t_n+h} f(x) \, dx = \frac{1}{2h} \int_{t-h}^{t+h} f(x) \, dx = f_h(t).$$

Since $C[0,1] \subset L^p[0,1]$, the lemma is proved. □

Lemma 3. *Let $1 \le p < \infty$, and $f \in L^p[0,1]$. Then for each $h > 0$, we have*

$$|f_h(t)| \le (2h)^{-1/p} \, \|f\|_p, \qquad \text{for all } t \in [0,1]. \tag{1}$$

Proof. Assume $p > 1$. Let q be the conjugate index such that $1/p + 1/q = 1$. We apply Hölder's inequality to get

$$\begin{aligned} |f_h(t)|^p &= \frac{1}{(2h)^p} \left| \int_{t-h}^{t+h} 1 \cdot f(x)\, dx \right|^p \\ &\leq \frac{1}{(2h)^p} \left(\int_{t-h}^{t+h} 1\, dx \right)^{p/q} \cdot \int_{t-h}^{t+h} |f(x)|^p\, dx \\ &= \frac{1}{2h} \int_{t-h}^{t+h} |f(x)|^p\, dx. \end{aligned}$$

Thus we have shown

$$|f_h(t)|^p \leq \frac{1}{2h} \int_{t-h}^{t+h} |f(x)|^p\, dx \tag{2}$$

holds for $1 < p < \infty$ and for all $t \in [0, 1]$. When $p = 1$, (2) is obviously true. The inequality (1) follows from (2). □

Lemma 4. *With the notation of the last lemma, we have*

$$\| f_h \|_p \leq \| f \|_p \,. \tag{3}$$

Proof. Using the inequality (2), it follows that

$$\begin{aligned} \int_0^1 |f_h(t)|^p\, dt &\leq \frac{1}{2h} \int_0^1 \left[\int_{t-h}^{t+h} |f(x)|^p\, dx \right] dt \\ &= \frac{1}{2h} \int_0^1 \left[\int_{-h}^{h} |f(t+y)|^p\, dy \right] dt. \end{aligned} \tag{4}$$

Observe that $(t, y) \mapsto f(t + y)$ is Lebesgue measurable. Since the integrand is nonnegative, we can apply Fubini-Toneli's theorem to get

$$\begin{aligned} \int_0^1 \left[\int_{-h}^{h} |f(t+y)|^p\, dy \right] dt &= \int_{-h}^{h} \left[\int_0^1 |f(t+y)|^p\, dt \right] dy \\ &\leq 2h \int_0^1 |f(x)|^p\, dx. \end{aligned}$$

Thus (4) implies

$$\int_0^1 |f_h(t)|^p\, dt \leq \int_0^1 |f(x)|^p\, dx,$$

and the result follows. □

Theorem 5. *Let $1 \leq p < \infty$. Let $\mathcal{K}$ be a closed and bounded subset of $L^p[0, 1]$. Then $\mathcal{K}$ is compact iff for each $\varepsilon > 0$ there exists a $\delta > 0$ such that $\| f - f_h \|_p < \varepsilon$ for all $f \in \mathcal{K}$ and $0 < h < \delta$.*

Proof. We prove that the condition is necessary. Let $\varepsilon > 0$ be given. Since $C[0, 1]$ is dense in $L^p[0, 1]$ and $\mathcal{K}$ is compact there exists continuous functions f_j, $1 \leq j \leq n$, such that $\mathcal{K} \subset \cup_{j=1}^n B(f_j, \varepsilon)$.

By uniform continuity of each f_j, there exists some $\delta > 0$ such that $|f(x) - f(t)| < \varepsilon$ holds for each $1 \le j \le n$ for all $x, t \in [0,1]$ with $|x - t| < \delta$. In particular,, for $0 < h < \delta$, we have

$$|f_j(t) - (f_j)_h(t)| = \frac{1}{2h} | \int_{t-h}^{t+h} [f_j(t) - f_j(x)]\, dx| \le \varepsilon.$$

Thus, $\| f_j - (f_j)_h \|_p \le \varepsilon$.

Now if $f \in \mathcal{K}$, we choose j so that $f \in B(f_j, \varepsilon)$. By Lemma 4 we have $\| f_h - (f_j)_h \|_p \le \| f - f_j \|_p < \varepsilon$. Therefore,

$$\| f - f_h \|_p \le \| f - f_j \|_p + \| f_j - (f_j)_h \|_p + \| (f_j)_h - f_h \|_p < 3\varepsilon$$

holds for all $f \in \mathcal{K}$ and $0 < h < \delta$.

We now prove that the condition is sufficient. Since $L^p[0,1]$ is complete, it is enough to prove that $\mathcal{K}$ is totally bounded. Towards this end, let $\varepsilon > 0$ be given. Fix an $h > 0$ such that $\| f - f_h \|_p < \varepsilon$ for all $f \in \mathcal{K}$. Let $M > 0$ be such that $\| f \|_p < M$ for all $f \in \mathcal{K}$. By Lemma 4, it follows that

$$|f_h(t)| \le M(2h)^{-1/p} =: C,$$

holds for all $t \in [0,1]$ and $f \in \mathcal{K}$. Set $\mathcal{K}_h := \{f_{hh} : f \in \mathcal{K}\}$. Here

$$f_{hh} := \frac{1}{2h} \int_{t-h}^{t+h} f_h(x)\, dx.$$

Clearly, $|f_{hh}(t)| \le C$ for $t \in [0,1]$ and $f \in \mathcal{K}$. Hence $\mathcal{K}_h$ is a bounded set in $(C[0,1] < \| \, \|_{\sup})$.

We now show that $\mathcal{K}_h$ is equicontinuous. To see this, note that if $s < t$ and $f \in \mathcal{K}$, then

$$\begin{aligned} |f_{hh}(t) - f_{hh}(s)| &= \frac{1}{2h} | \int_{t-h}^{t+h} f_h(x)\, dx - \int_{s-h}^{s+h} f_h(x)\, dx| \\ &= \frac{1}{2h} | \int_{s+h}^{t+h} f_h(x)\, dx - \int_{s-h}^{t+h} f_h(x)\, dx| \\ &\le \frac{1}{2h} \left[\int_{s+h}^{t+h} |f_h(x)|\, dx + \int_{s-h}^{t-h} |f_h(x)|\, dx \right] \\ &\le \frac{1}{2h} [2C(t-s)] = \frac{C}{h}(t-s). \end{aligned}$$

The equicontinuity of $\mathcal{K}_h$ follows from this.

By Arzela-Ascoli, $\mathcal{K}_h$ is totally bounded and hence we can choose $f_1, \ldots, f_n \in \mathcal{K}$ such that for each $f \in \mathcal{K}$, there exists j such that $\| f_{hh} - (f_j)_{hh} \|_{\sup} < \varepsilon$. We then have

$$\begin{aligned} \| f - f_j \|_p &\le \| f - f_h \|_p + \| f_h - f_{hh} \|_p + \| f_{hh} - f_j \|_p \\ &< 2\varepsilon + \| f_{hh} - f_j \|_p \\ &< 2\varepsilon + \| f_{hh} - (f_j)_{hh} \|_p + \| (f_j)_{hh} - (f_j)_h \|_p + \| (f_j)_h - (f_j)_h \|_p \\ &< 5\varepsilon. \end{aligned}$$

This shows that $\mathcal{K}$ is totally bounded subset of $L^p[0,1]$ and completes the proof of the theorem. □

We now characterize compact subsets of $L^p(\mathbb{R})$. We need to impose one more condition on the family, viz. that of 'uniform integrability' (see condition (iv) in the theorem below). We use $\| f \|$ in place of $\| f \|_p$ to simplify our typing.

Theorem 6. *Let $\mathcal{K} \subset L^p(\mathbb{R})$, $1 \le p < \infty$. Then $\mathcal{K}$ is compact iff $\mathcal{K}$ satisfies the following conditions:*

(i) *$\mathcal{K}$ is closed.*

(ii) *$\mathcal{K}$ is bounded.*

(iii) *Given $\varepsilon > 0$, there exists $\delta > 0$ such that*

$$\int_{\mathbb{R}} |f(t+s) - f(s)|^p \, ds < \varepsilon \quad \textit{forall} f \in \mathcal{K}, 0 < |t| < \delta.$$

(iv) *Given $\varepsilon > 0$, there exists $\alpha > 0$ such that*

$$\int_{|s|>\alpha} |f(s)|^p \, ds < \varepsilon, \qquad \textit{for all } f \in \mathcal{K}.$$

Proof. Let $\mathcal{K}$ be compact. Then it is closed and bounded. In fact. it is totally bounded. Therefore, given $\varepsilon > 0$, there exist $f_1, \ldots, f_n \in L^p$ such that $\mathcal{K} \subset \cup_j B(f_j, \varepsilon)$. Since the set of all finite linear combinations of characteristic functions of bounded intervals is dense in L^p, there exist such functions g_j such that $\| f_j - g_j \| < \varepsilon$. Now for $\alpha > 0$ sufficiently large, support of g_j will be contained in $[-\alpha, \alpha]$ for $1 \le j \le n$. Hence for all such large α, we have

$$\begin{aligned} \left[\int_{-\infty}^{-\alpha} |f(s)|^p + \int_{\alpha}^{\infty} |f(s)|^p \right]^{1/p} &\le \left[\int_{-\infty}^{-\alpha} |f(s) - g_j(s)|^p + \int_{\alpha}^{\infty} |f(s) - g_j(s)|^p \right]^{1/p} \\ &\quad + \left[\int_{-\infty}^{-\alpha} |g_j(s)|^p + \int_{\alpha}^{\infty} |g_j(s)|^p \right]^{1/p} \\ &\le \| f - g_j \| + \left[\int_{-\infty}^{-\alpha} |g_j(s)|^p + \int_{\alpha}^{\infty} |g_j(s)|^p \right]^{1/p}. \end{aligned}$$

Hence (iv) follows from

$$\| f - g_j \| \le \| f - f_j \| + \| f_j - g_j \| \le 2\varepsilon.$$

To prove (iii), we start by observing that it holds for the characteristic function χ_J of any finite interval J:

$$\lim_{t \to 0} \int_{\mathbb{R}} |\chi_J(s+t) - \chi_J(s)|^p \, ds = 0.$$

Thus (iii) holds for finite linear combinations of such functions, in particular, for g_j's as above, Hence we have for any $f \in \mathcal{K}$,

$$\begin{aligned}
\limsup_{t\to 0}\left(\int_{\mathbb{R}} |f(s+t)-f(s)|^p\,ds\right)^{1/p} &\le \limsup_{t\to 0}\left(\int_{\mathbb{R}} |f(s+t)-f_j(s+t)|^p\,ds\right)^{1/p} \\
&\quad + \limsup_{t\to 0}\left(\int_{\mathbb{R}} |f_j(s+t)-g_j(s+t)|^p\,ds\right)^{1/p} \\
&\quad + \limsup_{t\to 0}\left(\int_{\mathbb{R}} |g_j(s+t)-g_j(s)|^p\,ds\right)^{1/p} \\
&\quad + \left(\int_{\mathbb{R}} |g_j(s)-f_j(s)|^p\,ds\right)^{1/p} \\
&\quad + \left(\int_{\mathbb{R}} |f_j(s)-f(s)|^p\,ds\right)^{1/p} \\
&\le \varepsilon+\varepsilon+0+\varepsilon+\varepsilon,
\end{aligned}$$

if j is so chosen that $\| f - f_j \| < \varepsilon$. This completes the proof of the necessary part of the conditions.

To prove the sufficiency part, we need only show that if $\mathcal{K}$ satisfies the conditions, then $\mathcal{K}$ is totally bounded. We define the translation operator T_t by $T_t f(s) := f(t+s)$. Condition (ii) means that $T_t f \to f$ in L^p as $t \to 0$ *uniformly* for $f \in \mathcal{K}$. We also define the mean value

$$M_a f(s) := \frac{1}{2a}\int_{-a}^{a} T_t f(s)\,ds \equiv \frac{1}{2a}\int_{-a}^{a} f(t+s)\,ds.$$

Using Hölder's inequality and Fubini-Tonelli as in the last theorem, we get

$$\begin{aligned}
\| M_a f - f \| &\le \left[\int_{\mathbb{R}}\left(\int_{-a}^{a}\frac{1}{2a}|f(t+s)-f(s)|\,dt\right)^p ds\right]^{1/p} \\
&\le \frac{1}{2a}\left[\int_{\mathbb{R}}\int_{-a}^{a}|f(t+s)-f(s)|^p\,dt(2a)^{p/q}\,ds\right]^{1/p} \\
&\le \left(\frac{1}{2a}\int_{-a}^{a}dt\int_{\mathbb{R}}|f(t+s)-f(s)|^p\,ds\right)^{1/p}.
\end{aligned}$$

Thus we have shown

$$\| M_a f - f \| \le \sup_{|t|\le a} \| T_t f - f \|. \tag{5}$$

Hence it is enough to establish the total boundedness of $\mathcal{K}_a := \{M_a f : f \in \mathcal{K}\}$ for any fixed but sufficiently small a.

We shall show that for any fixed $a > 0$, the set $\mathcal{K}_a$ is bounded in L^∞-norm and equicontinuous. In fact, we have

$$\begin{aligned}
|M_a f(s_1) - M_a f(s_2)| &\le \frac{1}{2a}\int_{-a}^{a}|f(s_1+t)-f(s_2+t)|\,dt \\
&\le \left[\frac{1}{2a}\int_{-a}^{a}|f(s_1+t)-f(s_2+t)|^p\,dt\right]^{1/p}.
\end{aligned}$$

This along with (ii) shows that $\mathcal{K}_a$ is equicontinuous. A similar proof establishes that $\mathcal{K}_a$ is bounded in L^∞-norm.

Let $\alpha > 0$ be given. By Arzela-Ascoli applied to $C[-\alpha, \alpha]$ for any given $\varepsilon > 0$, there exist $M_a f_j$, $1 \leq j \leq n$ with $f_j \in \mathcal{K}$ such that for any given $f \in \mathcal{K}$, there exists j such that

$$\sup_{|s| \leq \alpha} |M_a f(s) - M_a f_j(s)| \leq \varepsilon. \tag{6}$$

We use this to show that $\mathcal{K}_a$ is totally bounded in $L^p(\mathbb{R})$.

We have

$$\begin{aligned} \| M_a f - M_a f_j \|^p &= \int_{-\alpha}^{\alpha} |M_a f(s) - M_a f_j(s)|^p \, ds \\ &\quad + \int_{|s|>\alpha} |M_a f(s) - M_a f_j(s)|^p \, ds. \end{aligned} \tag{7}$$

The first term on the right side is $\leq 2\alpha\varepsilon^p$ for an appropriate choice of j. We estimate the second term on the right of (7):

$$\begin{aligned} \int_{|s|>\alpha} |M_a f(s) - M_a f_j(s)|^p \, ds &\leq \| M_a f - f \| + \left(\int_{|s|>\alpha} |f(s) - f_j(s)|^p \, ds \right)^{1/p} \\ &\quad + \left(\int_{|s|>\alpha} |f_j(s) - M_a f_j(s)|^p \, ds \right)^{1/p}. \end{aligned}$$

The term $\| M_a f - f \| \to 0$ as $a \to 0+$. By virtue of (iii), the other two terms go to 0 as $\alpha \nearrow \infty$ if a remains bounded. This completes the proof of the fact that $\mathcal{K}_a$ is totally bounded in $L^p(\mathbb{R})$ and hence the proof of the theorem. □