# The Role of LUB Axiom in Real Analysis

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The aim of this article is to bring out the decisive role played by the so-called *Least upper bound property* of real numbers. Most of the students — even the best ones — do not realize the importance of this axiom at the end of a year long course in real analysis. The removal of the single piece, viz., the LUB axiom will cause the entire edifice of real analysis to collapse. Most often the major results use other results which in turn depend either directly or indirectly on the LUB property of the reals. This obscures the significance of the axiom. In this article I shall try to show as directly as possible how LUB property enters the proof of major results in real analysis. I have written the article keeping an average undergraduate of Indian Universities in mind. It is my hope that teachers would give copies of this article to good students and ask them to present it as seminars. The experts may find the details a bit too excessive.

Let us recall what the LUB property of reals is. It is formulated as follows:

*Let $A$ be a nonempty subset of reals. Assume that $A$ is bounded above — in the sense that there is an $M \in \mathbb{R}$ such that $a \leq M$ for all $a \in A$. Then there exists an $\alpha \in \mathbb{R}$ with the property that i) $a \leq \alpha$ for all $a \in A$ and ii) if $\beta$ is any upper bound for $A$ then $\alpha \leq \beta$.*

This number $\alpha$ is unique and called the least upper bound for $A$. We shall denote it by $\alpha = \sup A$. The following easy exercise gives us the most useful characterization of $\sup A$.

**Ex. 1.** Let $A$ be a nonempty subset of reals bounded above. A real number $\alpha$ is $\sup A$ iff the following hold: i) $\alpha$ is an upper bound for $A$. ii) If $\varepsilon > 0$ is given then there is an $a \in A$ such that $\alpha - \varepsilon < a$.

To keep things in perspective let us recall that $\mathbb{Q}$ is an ordered field which does not enjoy the LUB property. (See the following exercise.) We shall repeatedly use this fact to construct examples to show how our results fail if we consider $\mathbb{Q}$ in place of $\mathbb{R}$. To understand the remarks completely that follow the theorems the beginner may need the guidance of a teacher. I also would like to point out that the examples are *external* rather than intrinsic. This is for two reasons: i) The average beginner may not be able to appreciate the excessively pedantic intrinsic examples. ii) The students may thus appreciate the fact that how the lack of "holes" in $\mathbb{R}$ helps one prove better results.

**Ex. 2.** Let $A := \{x \in \mathbb{Q} : x \geq 0 \ \& \ x^2 \leq 2\}$. Then $A$ is bounded above and it has no least upper bound **in** $\mathbb{Q}$. *Hint:* If $\alpha \in \mathbb{Q}$ is $\sup A$, then by trichotomy one of the following holds:

i) $\alpha^2 = 2$, ii) $\alpha^2 < 2$ or iii) $\alpha^2 > 2$. That the first case is impossible is a well-known fact. In the other cases show that there exists $N \in \mathbb{N}$ such that $(\alpha + 1/N)^2 < 2$ or $(\alpha - 1/N)^2 > 2$. Arrive at a suitable contradiction. See the proof of Theorem 6.

**Theorem 3.** **[Nested Interval Theorem]** *Let $J_n := [a_n, b_n]$ be intervals in $\mathbb{R}$ such that $J_{n+1} \subseteq J_n$ for all $n \in \mathbb{N}$. Then $\cap J_n \neq \emptyset$.*

**Proof.** Note that the hypothesis means that $[a_{n+1}, b_{n+1}] \subset [a_n, b_n]$ for all $n$. In particular, $a_n \leq a_{n+1}$ and $b_{n+1} \leq b_n$ for all $n \in \mathbb{N}$. Let $E$ be the set of left endpoints of $J_n$. Thus, $E := \{a \in \mathbb{R} : a = a_n \text{ for some } n\}$. $E$ is nonempty.

We claim that $b_k$ is an upper bound for $E$ for each $k \in \mathbb{N}$, i.e., $a_n \leq b_k$ for all $n$ and $k$. If $k \leq n$ then $[a_n, b_n] \subseteq [a_k, b_k]$ and hence $a_n \leq b_n \leq b_k$. (Draw pictures!) If $k > n$ then $a_n \leq a_k \leq b_k$. Thus the claim is proved. By the LUB axiom there exists $c \in \mathbb{R}$ such that $c = \sup E$. We claim that $c \in J_n$ for all $n$. Since $c$ is an upper bound for $E$ we have $a_n \leq c$ for all $n$. Since each $b_n$ is an upper bound for $E$ and $c$ is the least upper bound for $E$ we see that $c \leq b_n$. Thus we conclude that $a_n \leq c \leq b_n$ or $c \in J_n$ for all $n$. Hence $c \in \cap J_n$. $\square$

**Remark 4.** This result is false in $\mathbb{Q}$. Here any interval $[a, b]$ for $a, b \in \mathbb{Q}$ is defined in the obvious way: $[a, b] := \{x \in \mathbb{Q} : a \leq x \leq b\}$. Consider an increasing (resp. decreasing) sequence $(a_n)$ (resp. $(b_n)$) of rational numbers converging to $\sqrt{2}$. Then the sequence $([a_n, b_n]$ of intervals in $\mathbb{Q}$ (consisting of rational numbers) and whose lengths go to 0 have empty intersection.

**Remark 5.** The usual form of Theorem 3 is as follows: *Let $(J_n := [a_n, b_n])$ be a sequence of nested closed and bounded intervals in $\mathbb{R}$. Assume that their lengths go to zero:* $\lim(b_n - a_n) = 0$. *Then $\cap J_n$ consists of a single point.*

This follows from our version. We have already shown the existence of a point in the intersection. Suppose that there are two such, say, $\alpha, \beta \in \cap J_n$. Then we have $|\alpha - \beta| \leq b_n - a_n$ for all $n$. Hence $|\alpha - \beta| = 0$ or $\alpha = \beta$.

**Theorem 6.** *Let $\alpha \in \mathbb{R}$ be nonnegative and $n \in \mathbb{N}$. The there exists a unique non-negative $x \in \mathbb{R}$ such that $x^n = \alpha$.*

*Proof.* The crucial part of the theorem is the existence of such an $x$. Uniqueness holds even in any ordered field. If $\alpha = 0$, the result is obvious, so we assume that $\alpha > 0$ in the following.

Look at Fig. We define

$$S := \{t \in \mathbb{R} : t \geq 0 \text{ and } t^n \leq \alpha\}.$$

Since $0 \in S$, we see that $S$ is not empty. It is bounded above. For, by Archimedean property of $\mathbb{R}$, we can find $N \in \mathbb{N}$ such that $N > \alpha$. We claim that $\alpha$ is an upper bound for $S$. If this is false, then there exists $t \in S$ such that $t > N$. But, then we have

$$t^n > N^n \geq N > \alpha,$$

a contradiction, since for any $t \in S$, we have $t^n \leq \alpha$. Hence w conclude that $N$ is an upper bound for $S$. Thus, $S$ is a nonempty subset of $\mathbb{R}$ which is bounded above. By the LUB property of $\mathbb{R}$, there exists $x \in \mathbb{R}$ such that $x$ is the LUB of $S$. We claim that $x^n = \alpha$.

Exactly one of the following is true: (i) $x^n < \alpha$, (ii) $x^n > \alpha$ and (iii) $x^n = \alpha$. We shall show that the first two possibilities do not arise. The idea is as follows. Look at Figure again. If Case (i) holds, that is, if $x^n < \alpha$, then it is geometrically clear that for $y$ very near to $x$ and greater than $x$, we must have $y^n < \alpha$. In particular, we can find a positive integer $k \in \mathbb{N}$ such that $(x + 1/k)^n < \alpha$. It follows that $x + 1/k \in S$. This is a contradiction, since $x$ is supposed to be an upper bound for $S$. In the second case, when $x^n > \alpha$, by similar considerations, we can find $k \in \mathbb{N}$ such that $(x - 1/k)^n > \alpha$. Since $x - 1/k < x$ and $x$ is the least upper bound for $S$, there exists $t \in S$ such that $t > x - 1/k$. We then see

$$t^n > (x - 1/k)^n > \alpha.$$

This again leads to a contradiction, since $t \in S$.

So, to complete the proof rigorously, we need only prove the existence of a positive integer $k$ in each of the first two cases.

Case (i): Assume that $x^n < \alpha$. For any $k \in \mathbb{N}$, we have

$$\begin{aligned} (x + 1/k)^n &= x^n + \sum_{j=1}^{n} \binom{n}{j} x^{n-j} (1/k^j) \\ &\leq x^n + \sum_{j=1}^{n} \binom{n}{j} x^{n-j} (1/k) \\ &= x^n + C/k, \quad \text{where } C := \sum_{j=1}^{n} \binom{n}{j} x^{n-j}. \end{aligned}$$

If we choose $k$ such that $x^n + C/k < \alpha$, that is, for $k > C/(\alpha - x^n)$, it follows that $(x+1/k)^n < \alpha$.

Case (ii): Assume that $x^n > \alpha$. For any $k \in \mathbb{N}$, we have $(-1)^j (1/k^j) > -1/k$ for $j \geq 1$. We use this below.

$$\begin{aligned} (x - 1/k)^n &= x^n + \sum_{j=1}^{n} \binom{n}{j} (-1)^j x^{n-j} (1/k^j) \\ &\geq x^n - \sum_{j=1}^{n} \binom{n}{j} x^{n-j} (1/k) \\ &= x^n - C/k, \quad \text{where } C := \sum_{j=1}^{n} \binom{n}{j} x^{n-j}. \end{aligned}$$

If we choose $k$ such that $x^n - C/k > \alpha$, that is, if we take $k > C/(x^n - \alpha)$, it follows that $(x - 1/k)^n > \alpha$.

We now show that if $x$ and $y$ are non-negative real numbers such that $x^n = y^n = \alpha$, then $x = y$. Look at the following algebraic identity:

$$(x^n - y^n) \equiv (x - y) \cdot [x^{n-1} + x^{n-2} y + \cdots + x y^{n-2} + y^{n-1}].$$

If $x$ and $y$ are nonnegative with $x^n = y^n$ and if $x \neq y$, say, $x > y$ then the left hand side is zero while both the factors in brackets on the right are strictly positive, a contradiction.

This completes the proof of the theorem. $\square$

**Remark 7.** The analogous result for the field of rational numbers is false. As is well-known, there exists no rational number $x$ such that $x^2 = 2$. Adapting th above proof shows that the non-empty subset $S := \{t \in \mathbb{Q} : t \geq 0 \text{ and } t^2 \leq 2\}$ bounded above has no least upper bound in $\mathbb{Q}$.

**Theorem 8.** *Any increasing sequence of real numbers bounded above is convergent. That is, if* $(x_n)$ *is a sequence in* $\mathbb{R}$ *such that* $x_n \leq x_{n+1}$ *and there exists* $M \in \mathbb{R}$ *such that* $x_n \leq M$ *for all* $n \in \mathbb{N}$*, then* $\lim x_n$ *exists.*

**Proof.** Let $E := \{x \in \mathbb{R} : x = x_n \text{ for some } n \in \mathbb{N}\}$ be the image of the sequence. (For example, if $x_n = (-1)^n 1$ then $E = \{\pm 1\}$. If $x_n = 1$ for all $n$ then $E = \{1\}$.) By assumption $E$ is nonempty and bounded above by $M$. By the LUB axiom there exists $\ell \in \mathbb{R}$ which is $\sup E$. We shall show that $\lim x_n = \ell$.

Let $\varepsilon > 0$ be given. As $\ell - \varepsilon$ is not an upper bound for $E$ there exists an $N$ such that $\ell - \varepsilon < x_N$. As the sequence is increasing we have $x_N \leq x_n$ for all $n \geq N$. We thus see that $\ell - \varepsilon < x_n \leq \ell < \ell + \varepsilon$ for all $n \geq N$. That is, $x_n \in (\ell - \varepsilon, \ell + \varepsilon)$ for $n \geq N$ or $\lim x_n = \ell$. $\square$

**Remark 9.** This result is false in $\mathbb{Q}$. For let a sequence $(x_n)$ be recursively defined as follows: $x_1 = 1$ and $x_{n+1} = \frac{1}{2}(x_n + \frac{2}{x_n})$. Then $(x_n)$ is bounded below and eventually decreasing. If it converges **in** $\mathbb{Q}$ then the limit is $\sqrt{2}$!

**Theorem 10.** *Any Cauchy sequence in* $\mathbb{R}$ *converges.*

**Proof.** Let $(x_n)$ be a Cauchy sequence in $\mathbb{R}$. Let $\delta > 0$ be arbitrary. There exists a positive integer $N = N(\delta)$ such that for all $m \geq N$ and $n \geq N$, we have $|x_n - x_m| < \delta/2$. In particular we have $|x_n - x_N| < \delta/2$. Or, equivalently,

$$x_n \in (x_N - \delta/2, x_N + \delta/2) \qquad \text{for all } n \geq N.$$

From this we make the following observations:
(i) For all $n \geq N$, we have $x_n > x_N - \delta/2$.
(ii) If $x_n \geq x_N + \delta/2$, then $n \in \{1, 2, \ldots, N-1\}$. Thus the set of $n$ such that $x_n \geq x_N + \delta/2$ is finite. We shall apply these two observations below for $\delta = 1$ and $\delta = \varepsilon$.

Let $S := \{x \in \mathbb{R} : \text{ there exists infinitely many } n \text{ such that } x_n \geq x\}$. We claim that $S$ is nonempty, bounded above and that $\sup S$ is the limit of the given sequence.

From (i), we see that $x_N - 1 \in S$. Hence $S$ is nonempty.

From (ii) it follows that $x_N + 1$ is an upper bound for $S$. That is, we claim that $y \leq x_N + 1$ for all $y \in S$. If this were not true, then there exists a $y \in S$ such that $y > x_N + 1$ and such that $x_n \geq y$ for infinitely many $n$. This implies that $x_n > x_N + 1$ for infinitely many $n$. This contradicts ii). Hence we conclude that $x_N + 1$ is an upper bound for $S$.

By the LUB axiom, there exists $\ell \in \mathbb{R}$ which is $\sup S$. We claim that $\lim x_n = \ell$. let $\varepsilon > 0$ be given. As $\ell$ is an upper bound for $S$ and $x_N - \varepsilon/2 \in S$ (by (i)) we infer that $x_N - \varepsilon/2 \leq \ell$. Since $\ell$ is the least upper bound for $S$ and $x_N + \varepsilon/2$ is an upper bound for $S$ (from (ii)) we see that $\ell \leq x_N + \varepsilon/2$. Thus we have $x_N - \varepsilon/2 \leq \ell \leq x_N + \varepsilon/2$ or

$$|x_N - \ell| \leq \varepsilon/2.$$

For $n \geq N$ we have

$$\begin{aligned} |x_n - \ell| &\leq |x_n - x_N| + |x_N - \ell| \\ &< \varepsilon/2 + \varepsilon/2 = \varepsilon. \end{aligned}$$

We have thus shown that $\lim_{n\to\infty} x_n = \ell$. □

**Remark 11.** This result is also patently false in $\mathbb{Q}$. See the last remark.

**Definition** A *positive decimal form* is a series of the form

$$a_0 + \frac{a_1}{10} + \frac{a_2}{10^2} + \cdots + \frac{a_n}{10^n} + \cdots$$

denoted by $a_0.a_1a_2\cdots a_n\cdots$, where $a_0 \in \mathbb{Z}^+$ and $a_n \in \{0, 1, \ldots, 9\}$ for each $n \in \mathbb{N}$.

**Theorem 12.** *Every positive decimal form converges to a positive real number. (We then say the decimal form represents the real number).*

**Proof.** Consider the positive decimal form $a_0.a_1a_2\cdots a_n\cdots$. Now

$$\begin{aligned} s_1 &= a_0 + \frac{a_1}{10} \leq a_0 + \frac{9}{10} \\ s_2 &= a_0 + \frac{a_1}{10} + \frac{a_2}{10^2} \leq a_0 + \frac{9}{10} + \frac{9}{10^2} \\ &\vdots \qquad\qquad \vdots \end{aligned}$$

By induction we have

$$\begin{aligned} s_n &\leq a_0 + \frac{9}{10}(1 + \frac{1}{10} + \cdots + \frac{1}{10^{n-1}}) \\ &\leq a_0 + \frac{9}{10}\left(\frac{1}{1 - \frac{1}{10}}\right) = a_0 + 1. \end{aligned}$$

Then $\{s_n\}$ is an increasing sequence of positive reals and is bounded above by $a_0 + 1$. Hence $\{s_n\}$ is convergent to a real number $a \in \mathbb{R}$. (We then say $a_0.a_1a_2\ldots a_n\ldots$ represents $a$.) □

The fact that we need the LUB property of the reals to show that the set of real numbers is uncountable is hardly appreciated by many students.

**Theorem 13.** *The set* $[0, 1]$ *is uncountable.*

**Proof.** If $[0, 1]$ is countable, since $[0, 1]$ is infinite, there exists a bijection $f : \mathbb{N} \to [0, 1]$. Let $z_n := f(n)$. We define two sequences $(x_n)$ and $(y_n)$ whose elements are defined recursively. Let $x_1$ be the $z_r$ where $r$ is the first integer such that $0 < z_r < 1$. Let $y_1$ be $z_s$ where $s$ is the first integer such that $x_1 < z_s < 1$. Assume that we have chosen $(x_i)_{i=1}^n$ and $(y_i)_{i=1}^n$ with the property

$$0 = x_0 < x_1 < x_2 < \cdots < x_n < y_n < y_{n-1} < \cdots < y_2 < y_1 < y_0 = 1.$$

We choose $x_{n+1}$ to be the $z_r$ where $r$ is the first integer such that $x_n < z_r < y_n$. Let $y_{n+1}$ be $z_s$ where $s$ is the first integer such that $x_{n+1} < z_s < y_n$. Clearly the set $\{x_n\} \subset [0, 1]$ is nonempty and bounded above. Let $x := \sup\{x_n\}$. Then it is easily seen that $x \in [0, 1]$ and that $x \neq z_n$ for $n \in \mathbb{N}$. □

**Theorem 14. [Intermediate Value Theorem]** *Let $f\colon [a,b] \subset \mathbb{R} \to \mathbb{R}$ be continuous. Assume that $f(a) < 0 < f(b)$. Then there exists $c \in (a,b)$ such that $f(c) = 0$.*

**Proof.** Draw some pictures. We wish to locate the "first" $c$ from $a$ such that $f(c) = 0$. Towards this end, we define $E := \{x \in [a,b] : f(y) \le 0 \text{ for } y \in [a,x]\}$.

Using the continuity of $f$ at $a$ for $\varepsilon = -f(a)/2$, we can find a $\delta > 0$ such that $f(x) \in (3f(a)/2, f(a)/2)$ for all $x \in [a, a+\delta)$. This shows that $a + \delta/2 \in E$. Since $E$ is bounded by $b$ there is $c \in \mathbb{R}$ such that $c = \sup E$. Clearly we have $a + \delta/2 \le c \le b$ and hence $c \in (a,b]$. We claim that $c \in E$ and that $f(c) = 0$.

Since $c - 1/n$ is not an upper bound for $E$ there is an $x_n \in E$ such that $c - 1/n < x_n \le c$. By sandwich lemma, $\lim x_n = c$. By continuity of $f$ at $c$ we have $f(x_n) \to f(c)$. As $f(x_n) \le 0$ for all $n$ we conclude that $f(c) \le 0$. This implies that $c < b$ and hence $c \in (a,b)$. If $f(c) \ne 0$ then $f(c) < 0$. Arguing as in the first part of the proof and using the fact that $a < c < b$, we can find a sufficiently small $\eta > 0$ such that $(c-\eta, c+\eta) \subset (a,b)$ and such that for $x \in (c-\eta, c+\eta)$ we have $f(x) \in (3f(c)/2, f(c)/2)$. As $\lim x_n = c$, there is an $N$ such that $x_N \in (c-\eta, c+\eta)$. But then we see that $f(x) < 0$ for $x \in [a, x_N] \cup (c-\eta, c+\eta/2]$. Hence $c + \eta/2 \in E$. This contradicts the fact that $c = \sup E$. Hence we conclude that $f(c) = 0$. $\square$

**Remark 15.** This result is not true in $\mathbb{Q}$. Consider the function $f\colon \{x \in \mathbb{Q} : 0 \le x \le 2\} \to \mathbb{Q}$ given by $f(x) = x^2 - 2$. Then $f(0) = -2 < 0 < 2 = f(2)$. But however there is no *rational* number in the interval at which $f$ assumes the value 0.

**Theorem 16.** *Let $f\colon [a,b] \subset \mathbb{R} \to \mathbb{R}$ be continuous. Then*

1. *$f$ is bounded.*
2. *Let $M := \sup\{f(x) : x \in [a,b]\}$ and $m := \inf\{f(x) : x \in [a,b]\}$. Then there exist points $c$ and $d$ in $[a,b]$ such that $f(c) = M$ and $f(d) = m$.*

**Proof.** Let $E := \{x \in [a,b] : f \text{ is bounded on } [a,x]\}$. The conclusion of the theorem is that $b \in E$.

Since $f$ is continuous at $a$, given $\varepsilon = 1$, there exists a $\delta_0 > 0$ such that $f(x) \in (f(a) - 1, f(a) + 1)$ for all $x \in [a, a+\delta_0)$. Thus we see that $|f(x)| \le |f(a)| + 1$ for $x \in [a, a+\delta_0/2]$. Hence $a + \delta_0/2 \in E$. Obviously $E$ is bounded by $b$. Let $c = \sup E$. Since $a + \delta_0/2 \in E$ we have $a \le c$. Since $b$ is an upper bound for $E$, $c \le b$. Thus $a \le c \le b$. We intend to show that $c \in E$ and $c = b$. This will complete the proof.

For any $n \in \mathbb{N}$, $c - 1/n$ is not an upper bound for $E$. Therefore there is an $x_n \in E$ such that $c - 1/n < x_n \le c$. Since $f$ is continuous at $c$, for $\varepsilon = 1$ there is a $\delta > 0$ such that $f(x) \in (f(c) - 1, f(c) + 1)$ for all $x \in (c-\delta, c+\delta) \cap [a,b]$. By Sandwich lemma, $x_n \to c$. But there exists an $N \in \mathbb{N}$ such that $x_N \in (c-\delta, c+\delta)$. Since $x_N \in E$ there is an $M$ such that $|f(x)| \le M$ for $x \in [a, x_N]$. Also $f$ is bounded by $|f(c)| + 1$ on $(c-\delta, c+\delta) \cap [a,b]$. From these two facts we conclude that

$$|f(x)| \le \max\{M, |f(c)| + 1\}, \qquad \text{for all } x \in [a, c + 1/2N] \cap [a,b].$$

This shows that $c \in E$. Note that the above argument shows also that $c + 1/2N \in E$ if $c \ne b$. This contradicts the fact that $c = \sup E$. Hence $c = b$. This proves 1).

To prove 2), we argue by contradiction. If there exists no $x \in [a,b]$ such that $f(x) = M$ then $M - f(x)$ is continuous at each $x \in [a,b]$ and $M - f(x) > 0$ for all $x \in [a,b]$. If we let $g(x) := 1/(M - f(x)$ for $x \in [a,b]$, then $g$ is continuous on $[a,b]$. By 1), there exists $A > 0$ such that $g(x) \leq A$ for all $x \in [a,b]$. But then we have, for all $x \in [a,b]$, $g(x) := \frac{1}{M-f(x)} \leq A$ or $M - f(x) \geq \frac{1}{A}$. Thus we conclude that $f(x) \leq M - (1/A)$ for $x \in [a,b]$. This contradicts our hypothesis that $M = \sup\{f(x) : x \in [a,b]\}$. We therefore conclude that there exists $c \in [a,b]$ such that $f(c) = M$. Arguing similarly we can find a $d \in [a,b]$ such that $f(d) = m$. $\square$

**Remark 17.** Consider the function $g = 1/f$ where $f$ is as in Remark 5. Then $g$ is continuous but not bounded on the closed and bounded interval $\{x \in \mathbb{Q} : 0 \leq x \leq 2\}$.

**Theorem 18. [Heine-Borel Theorem]** *If a closed and bounded interval in $\mathbb{R}$ is covered by a family of open intervals, then it is covered by finitely many open intervals from the given family.*

*More precisely, let $[a,b]$ be a closed and bounded interval in $\mathbb{R}$. Let $\{J_\alpha : \alpha \in I\}$ be a family of open intervals indexed by an indexing set $I$. Assume that $[a,b] \subseteq \cup_{\alpha \in I} J_\alpha$. Then there exist finitely many $\alpha_1, \ldots, \alpha_n \in I$ such that $[a,b] \subseteq \cup_{i=1}^n J_{\alpha_i}$.*

**Proof.** Let $E := \{x \in [a,b] \mid [a,x]$ is covered by finitely many $J_\alpha\}$. As $a \in [a,b] \subset \cup_{\alpha \in I} J_\alpha$, there exists $\alpha \in I$ such that $a \in J_\alpha$. Since $a \in J_\alpha$ and $J_\alpha$ is open there exists $\varepsilon > 0$ such that $(a - \varepsilon, a + \varepsilon) \subset J_\alpha$. Hence $[a, a + \varepsilon/2]$ is covered by the single element $J_\alpha$. Thus, $a + \varepsilon/2 \in E$ and hence $E \neq \emptyset$.

$E$ is a nonempty subset of $\mathbb{R}$ bounded by $b$. Hence there a real number $\beta$ which is the supremum of $E$.

We claim that $\beta \in E$ and that $\beta = b$. The claim proves the result. Suppose the claim is false.

Now $\beta \in [a,b]$: For any $n \in \mathbb{N}$, $\beta - 1/n$ is not an upper bound for $E$. Hence there exists $x_n \in E$ such that $\beta - 1/n < x_n$. Since $b$ is an upper bound for $E$ we see that $\beta - 1/n < x_n \leq \beta \leq b$. By Sandwich lemma, $\lim x_n = \beta \leq b$. Also, since $a + \varepsilon/2 \in E$, $\beta \geq a$. Thus $a \leq \beta \leq b$.

There exists $\alpha_0 \in I$ such that $\beta \in J_{\alpha_0}$. Hence we can find an $\varepsilon > 0$ such that $(\beta - \varepsilon, \beta + \varepsilon) \subseteq V$, as $J_{\alpha_0}$ is open. Assume that $\beta \neq b$. Then we may assume that $\varepsilon$ is so small that $(\beta - \varepsilon, \beta + \varepsilon) \subseteq [a,b]$. Since $\beta = \sup E$, $\beta - \varepsilon$ is not an upper bound of $E$. Thus, there exists $x \in E$, such that $\beta - \varepsilon < x \leq \beta$. Since $x \in E$, there exists finitely many $J_{\alpha_i}$, $1 \leq i \leq n$ such that $[a,x] \subseteq \cup_{i=1}^n J_{\alpha_i}$. But then $[a, \beta + \varepsilon/2] \subseteq \cup_{i=1}^n J_{\alpha_i} \cup J_{\alpha_0}$. Hence $\beta + \frac{\varepsilon}{2} \in E$, a contradiction since $\beta = \sup E$. Hence $\beta = b$.

Note also that the above argument proves that $\beta \in E$. $\square$

**Remark 19.** Take sequences $(r_n)$ and $(s_n)$ rationals such that $(r_n)$ increases and $(s_n)$ decreases to $\sqrt{2}$. Let $I_n := (-1, r_n)$ and $J_m := (s_n, 3)$. Then the closed and bounded interval $K$ **in** $\mathbb{Q}$ defined by $K := \{x \in \mathbb{Q} : 1 \leq x \leq 2\}$ is contained in the union $(\cup_n I_n) \cup (\cup_m J_m)$. However we can not find finitely many $I$'s and $J$'s whose union is $K$.

**Theorem 20. [Bolzano Weierstrass Theorem]** *Let $A$ be an infinite bounded subset of $\mathbb{R}$. Then there is a cluster point of $A$ in $\mathbb{R}$.*

**Proof.** This proof imitates that of Theorem 10.

Let $E := \{x \in \mathbb{R} : x \leq a$ for infinitely many $a \in A\}$. Let $M \in \mathbb{R}$ be such that $-M \leq a \leq M$ for all $a \in A$. It is obvious that $-M \in E$. We can easily show that $E$ is bounded by $M$. Hence there exists $\ell \in \mathbb{R}$ such that $\ell = \sup E$. We claim that $\ell$ is a cluster point of $E$. That is, we need to show that for any given $\varepsilon > 0$ there exists a point $a \in (\ell - \varepsilon, \ell + \varepsilon) \cap A$ other than $\ell$ itself.

Since $\ell - \varepsilon$ is not an upper bound for $E$ there is an $x \in E$ such that $\ell - \varepsilon < x$. Since $x \in E$ there exist infinitely many elements $a \in A$ such that $x \leq a$. Hence there exist infinitely many elements $a \in A$ such that $\ell - \varepsilon < a$. Also except for finitely many $a \in A$ we have $a < \ell + \varepsilon$. For, otherwise, for infinitely many elements $a \in A$ we have $a \geq \ell + \varepsilon$. But then $\ell + \varepsilon \in E$. This contradicts the fact that $\ell = \sup E$. Thus there exist infinitely many $a \in A$ such that $\ell - \varepsilon < a < \ell + \varepsilon$. (Prove this. See Remark 22 below.) In particular there is at least one $a \in A \cap (\ell - \varepsilon, \ell + \varepsilon)$ which is different from $\ell$. □

**Remark 21.** The image of the sequence in Remark 9 is an infinite unbounded set which has no cluster point in $\mathbb{Q}$.

**Remark 22.** Let $B := \{a \in A : \ell - \varepsilon < a\}$. Then $B$ is an infinite subset of $A$. Let $F$ be the finite set of elements $a \in A$ such that $a \geq \ell + \varepsilon$. Let $C := \{a \in A : a < \ell + \varepsilon\}$. Then $C = A \setminus F$. Hence $B \cap C$ is an infinite subset of $A$:

$$B \cap C = B \cap (A \cap F^c) = B \cap A \cap F^c = B \cap F^c = B \setminus F.$$

Thus every $a \in B$ satisfies $\ell - \varepsilon < a < \ell + \varepsilon$. Hence the set of all such $a \in A$ such that $\ell - \varepsilon < a < \ell + \varepsilon$ is an infinite set.

**Remark 23.** The usual undergraduate version *Any bounded sequence of reals has a convergent subsequence* follows from this version: If the image of the sequence is finite then there exists an $x \in \mathbb{R}$ such that $x = x_n$ for infinitely many $n \in \mathbb{N}$. These $n$'s give rise to a subsequence which converges to $x$. If the image of the sequence is infinite then it is a bounded infinite subset of $\mathbb{R}$. Let $x$ be a cluster point of this set. Let $x_{n_k} \in (x - 1/k, x + 1/k)$ be an element of the sequence chosen inductively so that $x_{n_{k+1}} \notin \{x_{n_1}, \dots, x_{n_k}\}$. The subsequence $(x_{n_k})$ then converges to $x$.

**Remark 24.** The proof of the fact that any continuous function on a closed and bounded interval is uniformly continuous uses either Heine-Borel theorem or the Bolzano-Weierstrass theorem.

**Remark 25.** In the theory of differentiation the single most basic result is Rolle's theorem: *Let $f\colon [a, b] \to \mathbb{R}$ be continuous and differentiable on $(a, b)$. Assume that $f(a) = f(b)$. Then there exists a point $c \in (a, b)$ such that $f'(c) = 0$.* All major results such as the mean value theorem, characterization of monotone differentiable functions in terms of the derivatives and the constancy of a differentiable function on an interval iff the derivative vanishes and Taylor's theorem follow from Rolle's theorem. A proof of Rolle's theorem uses Theorem 16.

**Remark 26.** In the theory of Riemann integration, even to make the definition of Riemann integrability we need the LUB property of $\mathbb{R}$.

**Theorem 27.** *$[a, b]$ is connected.*

**Proof.** Assume otherwise. We then can write $[a,b] = U \cup V$ where $U$ and $V$ are nonempty proper open subsets of $[a,b]$ with $U \cap V = \emptyset$. Without loss of generality assume that $a \in U$. We intend to show that $U = [a,b]$ so that $V = \emptyset$.

Consider $E := \{x \in [a,b] : [a,x] \subset U\}$. Since $a \in U$ and $U$ is open there exists an $\varepsilon > 0$ such that $[a,\varepsilon) \subset U$. Hence $[a,\varepsilon/2] \subset U$ or $a + \varepsilon/2 \in E$ so that $E \neq \emptyset$. $E$ is clearly bounded above by $b$. Thus by the LUB axiom there exists a real number $c \in \mathbb{R}$ which is $\sup E$. Note that $a \leq c \leq b$.

We claim that $c \in E$. For each $n \in N$, $c - 1/n$ is not an upper bound for $E$. We can therefore find $x_n \in E$ such that $c - 1/n < x_n \leq c$. Clearly $\lim x_n = c$. Since $x_n \in E$, $x_n \in U$. Since $U$ is closed in $[a,b]$ (with respect to the subspace topology) and $c \in [a,b]$, we see that $c = \lim x_n \in U$. Now $[a,c) = \cup[a, c-1/n] \subseteq \cup[a,x_n]$. As each of $[a,x_n] \subset U$ we see that $[a,c) \subset U$. This along with the fact that $c \in U$ allows us to conclude that $[a,c] \subset U$ and hence $c \in E$.

We now show that $c = b$. This will complete the proof. Since $c \in U$ and $U$ is open there exists an (relatively) open subset containing $c$ lying in $U$. If $c < b$, then there exists an $N \in \mathbb{N}$ such that $(c - 1/N, c + 1/N) \subset U$. This means that the set $[a, c + 1/2N] \subset [a,c] \cup (c - 1/N, c + 1/N) \subset U$. Thus $c + 1/2N \in E$. This contradicts the fact that $c = \sup E$. Therefore our assumption that $c < b$ is wrong. Thus $c = b$. $\square$

**Remark 28.** Let $J$ be as in Remark 19. Then $J = U \cup V$ where $U := \cup I_n$ and $V := \cup J_m$ is a disconnection of $J$.

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## Some Odds and Ends

In this section, we prove some more important results in $\mathbb{R}$ which use LUB but which are true for rational number field also

**Theorem 29** (Archimidean Property)**.**
*(i) The set of natural numbers is not bounded above in* $\mathbb{R}$.
*(ii) Given two real numbers* $x, y$ *with* $x > 0$, *there exists a positive integer* $n$ *such that* $nx > y$.

*Proof.* If $\mathbb{N}$ is bounded above, then let $\alpha \in \mathbb{R}$ be the least upper bound for $\mathbb{N}$. That is, we have

$$\begin{aligned} n &\leq \alpha \text{ for all } n \in \mathbb{N} \\ n+1 &\leq \alpha \text{ for all } n \in \mathbb{N} \\ n &\leq \alpha - 1 \text{ for all } n \in \mathbb{N}. \end{aligned}$$

We therefore conclude that $\alpha - 1$ is an upper bound for $\mathbb{N}$. This contradicts our assumption that $\alpha$ is the least upper bound for $\mathbb{N}$. This completes the proof of (i).

(ii) is an immediate consequence of (i). If no such $n$ exists, then $n \leq y/x$ for all $n$. In other words, $\mathbb{N}$ is bounded above by $y/x$, contradicting (i). $\square$

**Remark 30.** In fact, (i) and (ii) are equivalent. One can give a direct proof (ii) adapting that of (i). We leave this as an exercise to the student.

**Theorem 31** (Density of $\mathbb{Q}$ in $\mathbb{R}$)**.** *Given $x, y \in \mathbb{R}$ with $x < y$, there exists a rational number $r$ such that $x < r < y$.*

*Proof.* Assuming the existence of such an $r$, we write it as $r = m/n$ with $n > 0$. So, we have $x < m/n < y$, that is, $nx < y < ny$. Thus we are claiming that the interval $[nx, ny]$ contains an integer. It is geometrically obvious that a sufficient condition for an interval $J = [a, b]$ to have an integer in it is that its length $b - a$ should be greater than 1. This gives us an idea how to look for an $n$. We start with the proof.

Since $y - x > 0$, by Archimedian property, there exists $n \in \mathbb{N}$ such that $n(y - x) > 1$. We consider the set $S := \{k \in \mathbb{Z} : k \leq nx\}$. This is a nonempty subset of $\mathbb{R}$. For, if $S = \emptyset$, then it follows that $k > nx$ for all $k \in \mathbb{Z}$. From this, we get $-k < -nx$ for all $k \in \mathbb{Z}$. In particular, $-nx$ is an upper bound for $\mathbb{N}$. This contradiction shows that $S$ is nonempty. $S$ is bounded above by $nx$. Let $\alpha \in \mathbb{R}$ be the least upper bound for $S$. Since $\alpha - 1 < \alpha$ and $\alpha$ is the LUB of $S$, there exists $k \in S$ such that $k > \alpha - 1$. Hence $\alpha < k + 1$. Look at Figure??? Let $m := k + 1$. We claim that $m > nx$. For, otherwise, $m \leq nx$ and hence $m = k + 1 \in S$. Since $\alpha$ is an upper bound for $S$, we see that $k + 1 < \alpha$. It contradicts our choice of $k$. This proves $m > nx$. We also claim that $m < ny$. If false, then $m \geq ny$. Thus the interval $[nx, ny]$ of length greater than 1 is contained in $[k, k + 1]$. See Figure. To prove this analytically, we proceed as follows:

$$1 = (k + 1) - k = m - k \geq ny - nx = n(y - x) > 1.$$

Thus we conclude that $nx < m < ny$. Dividing the inequalities by $n$,we get the required result. $\square$

**Corollary 32.** *Let the assumptions be as in the last theorem. Then there exists an irrational number $z$ such that $x < z < y$.*

*Proof.* Use the last result to the pair $\sqrt{x}$, $\sqrt{y}$ to find a rational number $r$ such that $\sqrt{2}x < r < \sqrt{2}y$. Dividing the inequality by $\sqrt{2}$ yields $x < r/\sqrt{2} < y$. Since $r/\sqrt{2}$ is irrational the corollary follows. $\square$

**Proposition 33** (Greatest Integer Function)**.** *Let $x \in \mathbb{R}$. Then there exists a unique $m \in \mathbb{Z}$ such that $m \leq x < m + 1$.*

*Proof.* Let $S := \{k \in \mathbb{Z} : k \leq x\}$. As seen above, $S \neq \emptyset$. It is bounded above by $x$. Let $\alpha \in \mathbb{R}$ be its least upper bound. Then there exists $k \in S$ such that $k > \alpha - 1$. Since $k \in S$, $k \leq x$. We claim that $k + 1 > x$. For, if false, then $k + 1 \leq x$. Therefore, $k + 1 \in S$. Since $\alpha$ is an upper bound for $S$, we must have $k + 1 < \alpha$ or $k < \alpha - 1$. This contradicts our choice of $k$. Hence we have $x < k + 1$. The proposition follows if we take $m = k$. $\square$