Row Rank of a Matrix Equals its Column Rank

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Let $A = (a_{ij})$ be an $m \times n$ matrix over a field F. We denote by A_i, the i-th row of A: $A_i := (a_{i1}, \dots, a_{in})$. The j-th column of A is denoted by A'_j and is given by

$$A'_j = \begin{pmatrix} a_{1j} \\ \vdots \\ a_{mj} \end{pmatrix}.$$

We usually consider the row-vectors A_i as elements of the n-dimensional vector space F^n consisting of all row vectors with values in F, (or, as elements of $M_{1\times n}(F)$, the n-dimensional vector space consisting of matrices of size $1 \times n$ with entries in F).

Recall that the row rank of the matrix A is the number of elements in a maximal linearly independent subset of $\{A_i : 1 \le i \le m\}$. This is same as saying that the row rank of A is the dimension of the vector space spanned by the vectors A_i. This subspace is generally known as the row-subspace of the matrix A.

Similar considerations apply to the column vectors A'_j.

Ex. 1. Formulate the analogous concepts for the column vectors of A.

Let $A = BC$ be the product of two matrices B, of size $m \times r$ and C of size $r \times n$. Using a standard notation, we then have $a_{ij} = \sum_{k=1}^r b_{ik}c_{kj}$. So, the i-th row A_i of A is given by

$$\begin{aligned}
&(a_{i1}, \dots, a_{in}) \\
&= \left(\sum_{k=1}^r b_{ik}c_{k1}, \dots, \sum_{k=1}^r b_{ik}c_{kn} \right) \\
&= (b_{i1}c_{11} + b_{i2}c_{21} + \cdots + b_{ir}c_{r1}, \dots, b_{i1}c_{1n} + b_{i2}c_{2n} + \cdots + b_{ir}c_{rn}) \\
&= b_{i1}(c_{11}, \dots, c_{1n}) + \cdots + b_{ir}(c_{r1}, \dots, c_{rn}) \\
&= b_{i1}C_1 + \cdots + b_{ir}C_r,
\end{aligned}$$

where C_i stands for the i-th row of the matrix C. We thus observe see that the i-th row of A is a linear combination of the rows of C with coefficients from the i-th row of B.

Ex. 2. Formulate the analogous observation for the j-th column of A. Do not proceed further till you have solved this exercise!

The analogous observation for the column A'_j is obtained from the following:

$$\begin{aligned} A'_j := \begin{pmatrix} a_{1j} \\ \vdots \\ a_{mj} \end{pmatrix} &= \begin{pmatrix} \sum_{k=1}^r b_{1k}c_{kj} \\ \vdots \\ \sum_{k=1}^r b_{mk}c_{kj} \end{pmatrix} \\ &= \begin{pmatrix} b_{11}c_{1j} + \cdots + b_{1r}c_{rj} \\ \vdots \\ b_{m1}c_{1j} + \cdots + b_{mr}c_{mj} \end{pmatrix} \\ &= c_{1j} \begin{pmatrix} b_{11} \\ \vdots \\ b_{m1} \end{pmatrix} + \cdots + c_{rj} \begin{pmatrix} b_{1r} \\ \vdots \\ b_{mr} \end{pmatrix} \\ &= c_{1j}B'_1 + \cdots + c_{rj}B'_r. \end{aligned}$$

That is, the j-th column of A is a linear combination of the columns of B with coefficients in the j-th column of C.

What we have established is summarized as follows:

Lemma 3. *If $A = BC$ as above, then the row rank of A is less than or equal to the row-rank of C and the column rank of A is at most that of B.* □

On the other way around, if any collection of r row vectors $C_1, \ldots, C_r$ span the row space of A, an $r \times n$ matrix C can be formed by taking these vectors as its rows. Then the i-th row $A_i = b_{i1}C_1 + \cdots + b_{ir}C_r$ of A is a linear combination of the rows of C and $A = BC$, where $B = (b_{ij})$ is the $m \times r$ matrix whose i-th row $B_i = (b_{i1}, \ldots, b_{ir})$ is formed from the coefficients giving the i-th row of A as a linear combination of the r rows of C. Similarly, if any r column vectors span the column space of A, and B is the $m \times r$ matrix formed by these columns, then the $r \times n$ matrix C formed from the appropriate coefficients satisfies $A = BC$.

We summarize our findings in the form of a proposition.

Proposition 4. *If an $m \times n$ matrix $A = BC$, with B of size $m \times r$ and C of size $r \times n$, then*
(i) *the row space of A is the linear span of the rows of C.*
(ii) *the column space of A is the linear span of the columns of B.*
Consequently, the row-rank and column-rank of A are at most r. □

Such factorizations are always possible. Indeed, $A = I_m A$ is an example with $B = I_m, C = A$ and $r = m$.

Theorem 5. *The row rank and the column rank of a matrix A are equal.*

Proof. Let A be an $m \times n$ matrix. If $A = 0$, then the row and column rank of A are both 0. So, we assume that $A \neq 0$. Let r be the smallest positive integer such that there is an $m \times r$ matrix B and an $r \times n$ matrix C satisfying $A = BC$. (This is possible.) Thus the r rows of C

form a minimal spanning set of the row space of A and the r columns of B form a minimal spanning set of the column space of A. Since any minimal spanning set in a vector space is a basis of the vector space, it follows that r is the dimension of the row space as well as that of the column space. Hence, the row rank r of A is equal to the column rank r of A. $\square$

Remark 6. This remark is due to Professor M.I. Jinnah. For any nonzero $m \times n$ matrix A of row rank r, we can find an $m \times r$ matrix B and an $r \times n$ matrix C such that $A = BC$. By Lemma 3, we have

$$\text{Column Rank of } A \leq \text{Column Rank of } B \leq r = \text{Row Rank of } A.$$

Thus for any matrix A, we conclude that the column rank of A is less than or equal to the row rank of A. Applying this to the the transposed matrix A^t, we get the reverse inequality.

The ideas above lead to a simpler proof given below. I like this, since if this is properly displayed, will appeal visually and make the proof easier to rememebr.

Proof. Let $B_1, \ldots, B_r$ be the set of linearly independent rows of A. Let us write $B_i = (b_{i1}, \ldots, b_{ij}, \ldots, b_{in})$. Then any i-th row of A is a linear combination of B's. We write these linear combinations explicitly.

$$\begin{aligned}
(a_{11}, \ldots, a_{1n}) &= c_{11}(b_{11}, \ldots, b_{1j}, \ldots, b_{1n}) + \ldots + c_{1r}(b_{r1}, \ldots, b_{rj}, \ldots, b_{rn}) \\
&\vdots \\
(a_{i1}, \ldots, a_{in}) &= c_{i1}(b_{11}, \ldots, b_{1j}, \ldots, b_{1n}) + \ldots + c_{ir}(b_{r1}, \ldots, b_{rj}, \ldots, b_{rn}) \\
&\vdots \\
(a_{m1}, \ldots, a_{mn}) &= c_{m1}(b_{11}, \ldots, b_{1j}, \ldots, b_{1n}) + \ldots + c_{ir}(b_{r1}, \ldots, b_{rj}, \ldots, b_{nj})
\end{aligned}$$

Let us read these vertiaclly and write the j-th column of this array of equations. We get

$$\begin{aligned}
\begin{pmatrix} a_{1j} \\ a_{2j} \\ \vdots \\ a_{mj} \end{pmatrix} &= \begin{pmatrix} c_{11} \\ c_{21} \\ \cdots \\ c_{mj} \end{pmatrix} b_{1j} + \cdots + \begin{pmatrix} c_{1r} \\ c_{2r} \\ \cdots \\ c_{mr} \end{pmatrix} b_{rj}, \qquad \text{for } 1 \leq j \leq n \\
&= b_{1j}C_1 + \cdots + b_{rj}C_r, \qquad \text{say.}
\end{aligned}$$

That is, the columns are linear combinatirons of C_k, $1 \leq k \leq r$. Hence the maximum number of linearly independent columns is at most r, the row rank of A. Thus the column rank of A is less than or equal to the row rank of A. Starting with columns, we may prove that the row rank of A is less than or equal ot the column rank of A. Or, we observe that the column rank (respectively, row rank) of A^T is the row rank (respectively, column rank) of A. Thus we get

$$\text{row-rank of } A = \text{column rank of } A^T \leq \text{row rank of } A^T = \text{column rank of } A.$$

Hence both the ranks are equal. □