Maurer-Cartan Forms and Rigidity for Space Curves

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Let $SO(n, \mathbb{R}) \equiv SO(n)$ be the group of all orthogonal linear maps with determinant 1. A map $A : I \subset \mathbb{R} \to SO(n, \mathbb{R})$ is said to be once continuously differentiable or C^1 if we write $A(s) := (a_{ji})$ with respect to the standard basis, the functions $s \mapsto a_{ji}(s)$ are C^1. Such an A is thought of as a curve in $SO(n)$. Since $SO(n) \subset M(n, \mathbb{R}) \cong \mathbb{R}^{n^2}$ via the map $X := (x_{ji}) \mapsto (x_{11}, x_{12}, \ldots, x_{1n}, x_{21}, \ldots, x_{n1}, \ldots, x_{nn})$ the differentiation and integration of matrix valued functions are the same as the case of vector valued functions as indicated earlier, that is, they are carried out component-wise. In particular, for any C^1-function $B : (s, t) \to M(n, \mathbb{R})$, we have:

$$\int_s^t B(\sigma)d\sigma = (\int_s^t b_{11}(\sigma)d\sigma, \int_s^t b_{12}(\sigma)d\sigma, \ldots, \int_s^t b_{nn}(\sigma)d\sigma).$$

We then have the fundamental theorem of calculus: for any continuously differentiable function $A : I \to M(n, \mathbb{R})$ and any fixed point $s_0 \in I$,

$$A(s) = A(s_0) + \int_{s_0}^s A'(\sigma)d\sigma. \tag{1}$$

Ex. 1. Let $A, B : I \to M(n, \mathbb{R})$ be C^1-functions. Show that $(A \circ B)'(s) = (AB)'(s) = A'(s)B(s) + A(s)B'(s)$. Hint: If $C(s)$ is their product, then its entries are given by $c_{ij} = \sum_k a_{ik}b_{kj}$.

Example 2. The most important example for us arises out of the Frenet frames of a regular C^3-curve $c : I \to \mathbb{R}^3$ as follows. We assume as usual that $c'(s)$ and $c''(s)$ are linearly independent at each $s \in I$. We denote by $\mathbf{e}$ the standard ordered orthonormal basis $\{e_1, e_2, e_3\}$ of $\mathbb{R}^3$. We let $\mathbf{e}(s)$ stand for the Frenet frame at the point $c(s)$. Here, of course $e_1(s) = \mathbf{t}(s), e_2(s) = \mathbf{n}(s)$ and $e_3(s) = \mathbf{b}(s)$. Then there exists an orthogonal matrix $A(s)$ such that $\mathbf{e}(s) = A(s)\mathbf{e}$ for all $s \in I$. Recall that $A(s) = (a_{ji}(s))$ where $e_i(s) = \sum a_{ji}(s)e_j$. $A \in SO(3)$ since it takes the oriented basis $\mathbf{e}$ to the orthonormal basis $\mathbf{e}(s)$ with the same orientation. From the description of the entries of A and the section on the Frenet frame and formulas, we see that $A : I \to SO(3)$ is C^1 and hence an example of the required type. Thus given a curve c we "lifted" it to a curve $A : I \to SO(3)$.

We remark that $\mathbf{e}(s)$ and $\mathbf{e}$ can be considered as 3×3-matrices if we write their components as column vectors. With this understanding the equation $\mathbf{e}(s) = A(s)\mathbf{e}$ is an equation of matrices where the RHS is the product of the two matrices.

We differentiate $\mathbf{e}(s) = A(s)\mathbf{e}$ (or equivalently, the equation $e_i(s) = A(s)e_i$) to get

$$\mathbf{e}'(s) = A'(s)\mathbf{e} = A'(s)A^{-1}(s)\mathbf{e}(s).$$

Let $C(s) := A'(s)A(S)^{-1}$. In this notation the Frenet equation Eq. **??** can be recast as

$$\mathbf{e}'(s) = C(s)\mathbf{e}(s) \qquad \text{for } s \in I. \tag{2}$$

$C(s)$ is called the Cartan (or Maurer-Cartan) matrix of A or the curve c.

Motivated by this, even in the general case of any such A, we define the Maurer-Cartan form or the *Cartan matrix* of A to be the matrix $C_A(s) := A'(s)A(s)^{-1} = A'(s)A^{-1}(s)$. When there is no possible source of confusion, we write $C(s)$ in place of $C_A(s)$.

Lemma 3. *The Cartan matrix C of any C^1-curve A in $SO(n)$ is skew-symmetric.*

Proof. Since $A(s)$ is orthogonal, $A^{-1} = A^*$. Hence we differentiate $A(s)A^*(s) = I$ to get $A'(s)A^*(s) + A(s)A'(s)^* = 0$. Or, $A'(s)A^*(s) + (A'(s)A^*(s))^* = 0$ and hence the lemma. □

Now the rigidity result Theorem **??** can be reformulated and reproved in a more elegant way as follows:

Theorem 4. *Let $c_i : I \to \mathbb{R}^3$ be regular C^3-curves parameterized by their arc length. Assume that their Cartan matrices C_i are equal at all points $s \in I$. Then there exists a rigid motion of $\mathbb{R}^3$ taking c_1 to c_2.*

Proof. First of all note that our hypothesis is same as the equality of the functions: $\kappa_1(s) = \kappa_2(s)$, $\tau_1(s) = \tau_2(s)$ for all s and this is same as the hypothesis of Theorem **??**. We use the obvious notation below.

We can write $A_1(s) = A_2(s)B(s)$ for some (necessarily orthogonal matrix) $B(s)$. Since the Cartan matrices of A_i are equal, we must have

$$\begin{aligned}
C_1(s) &= C_{A_2(s)B(s)}(s) \\
&= (A_2(s)B(s))'\,(A_2(s)B(s))^{-1} \\
&= \left(A_2'(s)B(s) + A_2(s)B'(s)\right)\left(B^{-1}(s)A_2^{-1}(s)\right) \\
&= A_2'(s)A_2^{-1}(s) + A_2(s)B'(s)B^{-1}(s)A_2^{-1}(s) \\
&= C_2(s) + A_2(s)B'(s)B^{-1}(s)A_2^{-1}(s).
\end{aligned}$$

Hence we deduce that $A_2(s)B'(s)B^{-1}(s)A_2^{-1}(s) = 0$ or $B'(s)B^{-1}(s) = 0$ and so $B'(s) = 0$. That is, $B(s) = B$, a constant (orthogonal matrix).

The up-shot of the last paragraph is that the Frenet equations, Eq. **??** of the curves imply that the curves c_1 and $B \circ c_2$ have the same tangent vectors at all points s.

If c is any C^1 curve, then we have from the fundamental theorem of calculus, that

$$c(s) = c(s_0) + \int_{s_0}^{s} c'(\sigma)d\sigma.$$

Applying this to the curve $B \circ c_2$, where s_0 is an arbitrary point in I, we get

$$\begin{aligned} B \circ c_2(s) &= B \circ c_2(s_0) + \int_{s_0}^{s} B \circ c_2(\sigma) d\sigma \\ &= B \circ c_2(s_0) - c_1(s_0) + c_1(s_0) + \int_{s_0}^{s} c_1'(\sigma) d\sigma \\ &= B \circ c_2(s_0) - c_1(s_0) + c_1(s). \end{aligned}$$

Thus the curves $T \circ c_2$ and c_1 differ by the translation by $B \circ c_2(s_0) - c_1(s_0)$. □

Existence of Space Curves with the given Curvature and Torsion

We shall now see how this matrix formulation helps us to prove the Fundamental theorem on the existence and uniqueness of space curves with the given curvature and torsion. Given two functions $\kappa, \tau : I \to \mathbb{R}$, the problem is to find a C^3-curve $c : I \to \mathbb{R}^3$ such that its curvature and torsion are κ and τ. Given these function, we consider the associated Cartan matrix $C(s) := \begin{pmatrix} 0 & \kappa(s) & 0 \\ -\kappa(s) & 0 & \tau(s) \\ 0 & -\tau(s) & 0 \end{pmatrix}$. We then look for the solution of the matrix differential equation $A'(s)A(s)^* = C(s)$ with the initial condition $A(s_0) = I$. The matrix equation can be written as $A'(s)A(s)^{-1} = C(s)$ or $A'(s) = C(s)A(s)$.

General Existence Theorem of ODE

However the above approach will not be useful if we want to prove such an existence result for space curves. Here what we do is typical of many theorems in analysis. We start with an approximate "solution" of the given equation and then *iterate* it to get better and better approximate solutions which converge (fortunately!) to the desired solution. We thus investigate the existence of the solution of the matrix differential equation $A'(s) = C(s)A(s)$ with the initial condition $A(s_0) = I$ where $C : I \to M(k, \mathbb{R})$, the $(k \times k)$-matrices. We have the following fundamental

Theorem 5. *Let $C : I \to M(k, \mathbb{R})$ a continuous map be given on a closed and bounded interval I. Let $s_0 \in I$ be a fixed point. Then there exists a unique $A : I \to M(k, \mathbb{R})$ such that $A'(s) = C(s)A(s)$ with the initial value $A(s_0) = \mathrm{Id}$.*

Furthermore, if $C(s)$ is skew-symmetric for all $s \in I$, then $A(s) \in O(k)$ for all s.

Proof. In fact, we rather look into the equivalent problem of solving the integral equation

$$A(s) = \mathrm{Id} + \int_{s_0}^{s} C(\sigma)A(\sigma) d\sigma.$$

(The problems are equivalent by the fundamental theorem of calculus Eq. **??**.)

We need some preliminary facts. For $A \in M(n, \mathbb{R})$, we let $\|A\| := \max_{1 \le i,j \le k}\{|a_{ij}|\}$. Then it is easy to check that $\|AB\| \le k \|A\| \|B\|$. In particular, $\|A^2\| \le k \|A\|^2$. Since C is continuous, there exists a constant M such that $\|C(s)\| \le M$ for all $s \in I$. We also invite the reader to solve

Ex. 6. Let $A, A_n \in M(k, \mathbb{R})$ for $n = 1, 2, \ldots$. Then $\| A - A_n \| \to 0$ iff the matrix entries $a^n_{ij} \to a_{ij}$ as $n \to \infty$ for all $1 \leq i, j \leq k$.

(*We should remark that this is not the only possible "norm" on $M(n, \mathbb{R})$. In case, the reader knows about the operator norm, and if he uses it in place of the norm above, then there will be no factor of k in the inequality $\| AB \| \leq \| A \| \, \| B \|$.*)

To prove the existence part of the theorem, it is enough to show that for any $s_1, s_2 \in I$ with $s_1 < s_0 < s_2$ the approximate solutions A_n converge uniformly to a solution of the integral equation. Then the existence on I follows.

To attend to the details we let $A_0(s) = \mathrm{Id}$ and feed it into the integral equation $A(s) = \mathrm{Id} + \int_{s_0}^{s} C(\sigma)A(\sigma)d\sigma$. That is, we define

$$\begin{aligned} A_1(s) &= \mathrm{Id} + \int_{s_0}^{s} C(\sigma)A_0(\sigma)d\sigma \\ A_2(s) &= \mathrm{Id} + \int_{s_0}^{s} C(\sigma)A_1(\sigma)d\sigma \\ \vdots \quad &\vdots \quad \vdots \\ A_k(s) &= \mathrm{Id} + \int_{s_0}^{s} C(\sigma)A_{k-1}(\sigma)d\sigma. \end{aligned}$$

One easily shows by induction that

$$\| A_{n+1}(s) - A_n(s) \| \leq k^{n-1} M^n \frac{|s - s_0|}{n!} \leq k^{n-1} M^n \frac{|s_2 - s_1|}{n!}$$

The term on the RHS of the above estimate is the n-th term of the Taylor expansion of $(1/k)e^{kM|s_1 - s_2|}$ which goes to 0 as $n \to \infty$. Thus the approximate solutions $A_n(s)$ converge uniformly on $[s_1, s_2]$ to a continuous solution $A(s)$ of the integral equation. Since the right side of the integral equation is an indefinite integral of the continuous function A, the left side A is indeed differentiable. It is also a solution of the given matrix differential equation.

If A and B are two solutions of the ODE $A' = CA$ with the IC (initial condition) $A(s_0) = \mathrm{Id}$, then by repeating the above argument, we show that $\| A_n - B_n \| \to 0$ as $n \to \infty$.

Proof of the orthogonality of A is similar to that of Lemma 3: Differentiate A^*A to get $(A')^*A + A^*A' = (CA)^*A + A^*CA = A^*(-C)A + A^*CA = 0$. Hence A^*A is a constant which is the identity at $s = s_0$. Therefore A is orthogonal. If C is skew-symmetric, uniqueness can also be proved by adopting the argument in the proof of Theorem 4. □

From this it is easy to deduce the following existence and uniqueness theorem for space curves:

Theorem 7. *Let $\kappa, \tau : I \to \mathbb{R}$ be continuous. Assume that $\kappa(s) \neq 0$ for $s \in I$. Let $x_0 \in \mathbb{R}^3$ be given. Let $\{e_1(s_0), e_2(s_0), e_3(s_0)\}$ be an (ordered) orthonormal basis having the same orientation as that of the standard basis. Then there exists a unique unit speed C^3-curve $c : I \to \mathbb{R}^3$ such that its curvature and torsion are the given functions κ and τ.*

Proof. Take $n = 3$ in the above theorem. Let $\kappa : I \to \mathbb{R}$ and $\tau : I \to \mathbb{R}$ be given. Let $C(s)$ be the Cartan matrix defined at the beginning of this section. We fix a point $x_0 \in R^3$ and take the orthonormal frame at x_0 to be the standard frame. We then take $e_i(s) := A(s)e_i$ as the Frenet frames. In particular, $e_1(s)$ as tangent vectors so that the curve is given by

$$c(s) := \int_{s_0}^{s} e_1(\sigma)d\sigma.$$

It is now an easy exercise to show that c is a curve with the given curvature and torsion. But however the Frenet frame of c at $c(s_0)$ may not be $\{e_1(s_0), e_2(s_0), e_3(s_0)\}$. To remedy this, we consider $\tilde{c} := B \circ c$, where B is the orthogonal matrix (with determinant 1) which takes the standard basis to this basis. Then $\tilde{c}$ is the required one. □

The Case of Plane Curves

This section is more demanding than the earlier; however this will motivate the study of matrix differential equations and exponential of matrices etc. The serious students are very strongly urged to go through this section.

It is easier to solve the matrix equation $A' = CA$ in a more concrete way in the case of a plane curve and hence the analogue of Theorem 7 for plane curves. So assume that we are given a continuous function $k : I \to \mathbb{R}$ on an interval I. We wish to find a curve $c : I \to \mathbb{R}^2$ such that its curvature $\kappa_c(s) = k(s)$ on I. Now the corresponding Cartan matrix is given by $C(s) = \begin{pmatrix} 0 & k(s) \\ -k(s) & 0 \end{pmatrix} = k(s)\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = k(s)J$. Thus the matrix differential equation we wish to solve becomes $A'(s)A(s)^{-1} = k(s)J$. The whole thing smacks of "exponential" and hence we may try

$$A(s) := e^{J\int_{s_0}^{s} k(\sigma)d\sigma} = \begin{pmatrix} \cos\int_{s_0}^{s} k(\sigma)d\sigma & \sin\int_{s_0}^{s} k(\sigma)d\sigma \\ -\sin\int_{s_0}^{s} k(\sigma)d\sigma & \cos\int_{s_0}^{s} k(\sigma)d\sigma \end{pmatrix}.$$

More leisurely, we define $B(s) := \int_{s_0}^{s} C(\sigma)d\sigma = \left(\int_{s_0}^{s} k(\sigma)d\sigma\right) J$. Then $C(s)$ is the derivative of $B(s)$ and they commute with each other (only in this case and hence needs verification!). Therefore $(B^n)' = nB^{n-1}B' = nB^{n-1}C = nCB^{n-1}$ so that $(e^B)' = Ce^B$. Here the exponential of a matrix is defined by the infinite series $\exp(A) \equiv e^A := \sum_{n=0}^{\infty}(A^n/n!)$. For more on the exponential, see the appendix. Thus if we take $A(s) := e^{J\int_{s_0}^{s} k(\sigma)d\sigma}$, then we have solved the matrix differential equation $A'(s) = C(s)A(s)$.

Now to get the curve c from this easy: For, what the foregoing tells us is that the Frenet frame is uniquely determined if we fix any orthonormal frame $\{e_1(s_0), e_2(s_0)\}$ with the same orientation as the standard basis. The Frenet frame at $c(s)$ is given by $\{e_1(s), e_2(s)\} = A(s)\{e_1(s_0), e_2(s_0)\}$. In particular, if we fix an initial point $(x_0, y_0) \in \mathbb{R}^2$ as $c(s_0)$, then the tangent field is given by $e_1(s) = A(s)e_1(s_0)$ and hence the curve is given by $c(s) := c(s_0) + \int_{s_0}^{s} e_1(\sigma)d\sigma$. That is, the curve is given by

$$c(s) = (x_0 + \int_{s_0}^{s}(\cos\int_{s_0}^{s} k(\sigma)d\sigma), y_0 + \int_{s_0}^{s}(-\sin\int_{s_0}^{s} k(\sigma)d\sigma)).$$

Ex. 8. It is an instructive exercise to derive the expression for the plane curve by completely elementary means without the use of the matrix exponential.

Exercise: The Exponential Map in $M(n, \mathbb{R})$

The following set of exercises introduces the exponential map in $M(n, \mathbb{R})$ and its properties:

1) Show that if $f\colon U \subset \mathbf{E} \to \mathcal{F}$ is differentiable at x, then it remains so if $\mathbf{E}$ and F are endowed with equivalent norms. What is $f'(x)$ in this case?

2) For $X \in M(n, \mathbb{R})$, $X := (x_{ij})$, let

$$\| X \| := \max_{1 \le i,j \le n} |x_{ij}|$$

be the max norm. It is equivalent to the operator norm on elements of $M(n, \mathbb{R})$ *viewed as linear operators on* $\mathbb{R}^n$.

3) We have $\| AB \| \le n \| A \| \| B \|$ for all $A, B \in M(n, \mathbb{R})$ and $\left\| A^k \right\| \le n^{k-1} \| A \|^k$.

4) A sequence $A_k \to A$ in the max norm iff $a^k_{ij} \to a_{ij}$ for all $1 \le i, j \le n$ as $k \to \infty$. Here we have $A_k := (a^k_{ij})$ etc.

5) If $\sum_{k=0}^{\infty} \| A_k \|$ is convergent, then $\sum_0^\infty A_k$ is convergent to an element A of $M(n, \mathbb{R})$.

6) For any $X \in M(n, \mathbb{R})$, the series $\sum_0^\infty \frac{X^k}{k!}$ is convergent. We denote the sum by $\exp(X)$ or by e^X.

7) For a fixed $X \in M(n, \mathbb{R})$ the function $f(t) := e^{tX}$ satisfies the matrix differential equation $f'(t) = Xf(t)$, with the initial value $f(0) = I$. Hint: The (i, j)-th entry of $f(t)$ is a power series in t and use 4).

8) Set $g(t) := e^{tX} e^{-tX}$ and conclude that e^{tX} is invertible for all $t \in \mathbb{R}$ and for all $X \in M(n, \mathbb{R})$.

9) There exists a unique solution for $f'(t) = Af(t)$ with initial value $f(0) = B$ given by $f(t) = e^{tA}B$. Hint: If g is any solution consider $h(t) = g(t)\, e^{-tA}$.

10) Let A, $B \in M(n, \mathbb{R})$. If $AB = BA$ then we have $e^{A+B} = e^A e^B = e^B e^A = e^{B+A}$. Hint: Consider $\phi(t) := e^{t(A+B)} - e^{tA} e^{tB}$.

11) For A, $X \in M(n, \mathbb{R})$ we have $e^{AXA^{-1}} = Ae^X A^{-1}$.