Minkowski Inequality

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The standard proof of the Minkowski inequality is the deduce it from the Hölder's inequality. We give a direct proof below.

Lemma 1. *Let* $1 \leq p < \infty$. *Then*

$$(u+v)^p = \inf\{t^{1-p}u^p + (1-t)^{1-p}v^p : 0 < t < 1\} \text{ for all real } p, q \geq 0. \tag{1}$$

Proof. The nontrivial case is $p > 1$ and $u, v > 0$. Consider the function $f \colon (0,1) \to \mathbb{R}$ given by $f(t) := t^{1-p}u^p + (1-t)^{1-p}v^p$. It tends to infinity as t approaches the endpoints:

$$\begin{aligned}
\lim_{t\to 0_+} \frac{u^p}{t^{p-1}} + \frac{v^p}{(1-t)^{p-1}} &= \infty \\
\lim_{t\to 1_-} \frac{u^p}{t^{p-1}} + \frac{v^p}{(1-t)^{p-1}} &= \infty.
\end{aligned}$$

We find the minimum points of f using calculus. At the minimum point, we have

$$t^{-p}u^p = (1-t)^{-p}v^p. \tag{2}$$

It follows that $t = u/(u+v)$ so that $1 - t = v/(u+v)$. Using this information, we see the minimum value is as stated. □

Let $(X, \mathcal{B}, \mu)$ be a measure space. Let $f, g \in L^p(X)$ where $1 \leq p < \infty$. From the lemma we deduce that

$$|f+g|^p \leq (|f|+|g|)^p \leq t^{1-p}|f|^p + (1-t)^{1-p}|g|^p \text{ for } 0 < t < 1. \tag{3}$$

On integration,we obtain

$$\begin{aligned}
\|f+g\|^p &\leq t^{1-p}\|f\|^p + (1-t)^{1-p}\|g\|^p \text{ for } 0 < t < 1 \\
&\leq \inf_{0<t<1}\{t^{1-p}\|f\|^p + (1-t)^{1-p}\|g\|^p\} \\
&= (\|f\| + \|g\|)^p,
\end{aligned}$$

by the lemma. □

Remark 2. The reasoning of the lemma is known as a sub-convexity argument. Look at (1).