Vector Fields on Spheres *a la* Milnor

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A continuous vector field on S^{n-1} is a continuous map $F\colon S^{n-1} \to \mathbb{R}^n$. F is a tangent vector field if $F(p) \perp p = 0$ for all $p \in S^{n-1}$. We outline Milnor's proof of the non-existence of a nowhere vanishing continuous tangent vector field on even dimensional spheres.

Theorem 1. *Let n be odd. On S^{n-1} there does not exist any nowhere vanishing continuous (tangent) vector field.*

Proof. The following steps lead to a proof of this theorem.

Step 1. Let (X, d) be a compact metric space. Let $f\colon X \to Y$ be locally Lipschitz from X into another metric space Y. Then f is Lipschitz on X.

Step 2. Let $f\colon U \to \mathbb{R}^m$ be a C^1 map from an open set U in $\mathbb{R}^n$. Let K be a compact set in U. Then $f\colon K \to \mathbb{R}^m$ is Lipschitz. (Use the mean value theorem and Step 1.)

Step 3. Let U be an open connected bounded set in $\mathbb{R}^n$ so that $A = \overline{U}$ is compact and connected. Let F be a continuously differentiable vector field in an open set $V \supset A$. For $t \in \mathbb{R}$, let $F_t(x) := x + tF(x)$, for $x \in A$. If t is sufficiently small, then the mapping F_t is one-to-one and maps A onto $F_t(A)$ whose volume is a polynomial function of t. (Use change of variable formula.)

Step 4. Assume that $F\colon S^{n-1} \to \mathbb{R}^n$ be a C^1 tangent vector field on the sphere. If the parameter is sufficiently small, then F_t maps the unit sphere in $\mathbb{R}^n$ onto the sphere of radius $\sqrt{1+t^2}$. (Extend F to an annular region around S^{n-1}: $F(rx) := rF(x)$ for $a \leq r \leq$ and $x \in S^{n-1}$. Use inverse function theorem and a connectedness argument.)

Step 5. Given a C^1 field F of unit tangent vectors on S^{n-1}, we consider any annular region $a \leq \|x\| \leq b$ and extend F to this region as in the last lemma. Then F_t maps the sphere of radius r *onto* the sphere of radius $r\sqrt{1+t^2}$, for t near 0. Hence F_t maps the region A onto the annular region between the spheres of radii $a\sqrt{1+t^2}$ and $b\sqrt{1+t^2}$. Obviously, the volume of the latter region is given by

$$\text{Volume of } F_t(A) = (\sqrt{1+t^2})^n \text{Volume of } A.$$

If n is odd the volume of $F_t(A)$ is not a polynomial function of t.

Step 6. Extend the result to continuous vector fields. (By Stone-Weierstrass theorem, find P, a polynomial function, such that $\|F - P\|$ is less than half the minimum of F on the sphere. Get a tangent vector field from P by the obvious method. Show that it is nonzero.) $\square$