Bishop-Stone-Weierstrass Theorem

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Abstract

Gives a proof of Machado's version of Bishop-Sone-Weierstrass theorem.

Let X be a compact Hausdorff topological space. Let $\mathcal{A} \subset \mathcal{C}(X)$ be a subalgebra of $\mathcal{C}(X) \ni 1$. A set $E \subset X$ is said to be A-*antisymmetric* if $\forall f \in \mathcal{A}, \quad f$ is real valued on E then f is constant on E. The theorem of the title is

Theorem 1. *Let $\mathcal{A}$ be a closed subalgebra of $\mathcal{C}(X) \ni 1$. If $f \in \mathcal{C}(X)$ and if for every A-antisymmetric set $E \subset X, \quad \exists\, g \in \mathcal{A}$* such that $f = g$ on E *then $f \in \mathcal{A}$.*

The following are the various steps that lead to a very simple proof of this result.

1. Give some examples of A-antisymmetric sets.(You can choose various subalgebras A.)

2. Let A be closed and conjugate closed i.e.,$\mathcal{A} = \overline{\mathcal{A}}$. Assume that A separates points of X. What are the A-antisymmetric subsets?

3. If A is as in 1) show that the Stone-Weierstrass theorem is a consequence of B-S-W theorem.

4. $\forall f \in \mathcal{C}(X)$ and $E(\neq \emptyset) \subset X$, define
$$\| f \|_E := \sup\{| f(x) | : x \in E\}$$
and
$$d_f(E) := \inf\{\| f - g \|_E : g \in \mathcal{A}\}.$$
Then if $S(\neq \emptyset) \subset \ E \Rightarrow d_f(S) \leq \ d_f(E)$.

5. **Theorem**(Machado): Let $f \in \mathcal{C}(X)$. Then there exists a closed A- antisymmetric subset E of X such that $d_f(E) = d_f(X)$.
Prove that Machado's theorem implies B-S-W theorem.
The next few are steps towards a simple proof of Machado's result.

6. Let $\mathcal{F}$ be the family of all nonempty closed subsets E of X such that $d_f(E) = d_f(X)$. Then $\mathcal{F}$ is nonempty. If $\mathcal{C}$ is any totally ordered (with respect to inclusion) chain then it has a lower bound. Hence $\mathcal{F}$ has a minimal element say E.

Claim: E is an A-antisymmetric set.

The following leads to a proof of this claim.

7. If the claim is false then there is a function $h\colon\ E \to\ I\!R,\quad h \neq$ a constant on E. We may assume that
$$\min_E h(x) = 0 \qquad \max_E h(x) = 1$$
Define
$$E_1 = \{x \in E:\ 0 \leq h(x) \leq \tfrac{2}{3}\}$$
$$E_2 = \{x \in E:\ \tfrac{1}{3} \leq h(x) \leq 1\}$$
Then $E_i(\neq \emptyset) \subset E$ and $E_i \neq E$. There exists $g_i \in \mathcal{A}$ such that $\| f - g_i \|_{E_i} < d_f(X)$.

8. Define $h_n = (1 - h^n)^{2^n}$ and $f_n = h_n g_1 + (1 - h_n) g_2$. Then $h_n,\ f_n \in \mathcal{A}$ and $0 \leq h_n \leq 1$ on E.

9. $\| f - g_n \|_{E_1 \cap\ E_2} < d_f(X)$

10. $f_n \to\ g_1$ uniformly on $E_1 \setminus\ E_2$ and $f_n \to\ g_2$ uniformly on $E_2 \setminus\ E_1$.

11. For $n \gg\ 0$ we have $\| f - f_n \|_E < d_f(X)$ — a contradiction.

12. Extend Machado's result to vector valued functions.

13. Notice that Zorn's lemma can be avoided if we assume X is metrizable.

I refer the reader to the original paper:

Machado: *On Bishop's extension of the Stone-Weierstrass theorem,* Indag. *Math.* vol. 39 (1977) 218-224.

which also contains an elementary but a longer proof of Machado's theorem.