Miscellaneous Results in Analysis

S. Kumaresan
School of Math. and Stat.
University of Hyderabad
Hyderabad 500046
kumaresa@gmail.com

This article is a collection of results with possibly cute proofs. As I gathered them, I wrote them up lest that I forget such arguments!

Theorem 1 (AM-GM Inequality). *Let $x_1, x_2, \ldots, x_n$ be nonnegative real numbers. Their arithmetic mean $A := (x_1 + \cdots + x_n)/n$ is greater than or equal to their geometric mean $(x_1 \cdots x_n)^{1/n}$.*

Equality holds iff all x_i are equal.

Proof. We prove the result by induction. The result is true for $n = 2$. For,

$$(x_1 + x_2)^2 - 4x_1x_2 = (x_1 - x_2)^2 \geq 0.$$

It follows that $\frac{x_1+x_2}{2} \geq (x_1x_2)^{1/2}$. Observe that equality holds iff $(x_1 - x_2)^2 = 0$, that is, iff $x_1 = x_2$.

Assume that the result is true for any set $n-1$ nonnegative elements. Let $x_1, \ldots, x_n$ be nonnegative. Let $a := x_1 + \cdots + x_{n-1}$ and $b := x_1 \cdots x_{n-1}$. Then by induction hypothesis, $a/(n-1) \geq b^{1/(n-1)}$. We need to show that $(a + x_n) \geq n^n b x_n$.

Consider $f(x) := (a+x)^n - n^n bx$. Clearly, f is infinitely differentiable. We apply the derivative tests for extrema of this function. We find

$$f'(x) = n(a+x)^{n-1} - n^n b,$$

so that

$$f'(x) = 0 \iff (a+x)^{n-1} = n^{n-1}b \iff a + x = nb^{1/(n-1)} \iff x = b^{1/(n-1)} - a.$$

Let $c := b^{1/(n-1)} - a$. Then $f''(c) = n(n-1)(nb^{1/(n-1)})^{n-2} > 0$. Thus, c is a minimum for f. What is $f(c)$? We have

$$\begin{aligned} f(c) &= (nb^{1/(n-1)})^n - n^n b(nb^{1/(n-1)} - a) \\ &= n^n(b^{n/(n-1)} - nb^{n/(n-1)} + ab) \\ &= bn^n[b^{1/(n-1)}(1-n) + a] \\ &\geq 0, \end{aligned}$$

since by induction hypothesis $a \geq (n-1)b^{1/(n-1)}$. Also, $f(c) = 0$ iff $b^{1/(n-1)}(1-n) + a = 0$. The latter happens iff $x_1 = \cdots = x_{n-1} = x$, say, by induction hypothesis. But then $c = nx - (n-1)x = x$. Thus we have $x_n = x$. $\square$

Lemma 2. *Let $S := \{n + m\sqrt{2} : n, m \in \mathbb{Z}\}$. Let $a, b \in \mathbb{R}$ be such that $a < b$. Then there exists an $s \in S$ such that $a < s < b$. In other words, S is dense in $\mathbb{R}$.*

Proof. If $x, y \in S$ and $k \in \mathbb{Z}$, then $x \pm y, kx \in S$. Let $n(m) := [m\sqrt{2}]$, the greatest integer less than or equal to $m\sqrt{2}$. Then, $0 \leq m\sqrt{2} - n(m) < 1$.

It is easy to see that if $n + m\sqrt{2} = n' + m'\sqrt{2}$, then $n = n'$ and $m = m'$.

Let $s_m := m\sqrt{2} - n(m)$. Then $0 \leq s_m < 1$ and $s_m \in S$. Also, if $m \neq m'$, then $s_m \neq s_{m'}$. Hence we conclude that $\{s_m : m \in \mathbb{Z}\}$ is an infinite subset of $S \cap [0, 1)$.

Given $\varepsilon > 0$, we partition $[0, 1)$ into k equal parts so that each subinterval has length less than ε. At least one of these subintervals must contain two distinct elements, say, $s_m, s_{m'}$ of $S \cap [0, 1)$. Without loss of generality let us assume that $s_m < s_{m'}$. Then we have $0 < s_{m'} - s_m < \varepsilon$. Since $s_{m'} - s_m \in S$, we have shown that given $\varepsilon > 0$, there exists an element $s \in S$ with $0 < s < \varepsilon$.

Now, let $\varepsilon > 0$ such that $b - a > \varepsilon$ be given. Then there exists $n \in \mathbb{Z}$ such that $a < n\varepsilon < b$. For, choose n to be the least integer k such that $k\varepsilon > a$. Then $(n-1)\varepsilon \leq a < n\varepsilon$. We claim that $n\varepsilon < b$. For, otherwise,

$$b - a \leq n\varepsilon - (n-1)\varepsilon = \varepsilon,$$

a contradiction.

We take $\varepsilon := (b - a)/2$. Then there exists $s \in S$ such that $0 < s < \varepsilon$. Hence there exists an integer n such that $a < ns < b$. Since $ns \in S$, the theorem is proved. □

Lemma 3. *Let K be any bounded subset of $\mathbb{R}^n$. Then K is totally bounded.*

Proof. This statement is an avatar of Archimidean property of $\mathbb{R}$. Make sense out of this in the case of $n = 1$. The following proof owes its genesis to this idea.

In the general case, it is enough to show that any ball of the form $B(0, R)$ is totally bounded. For, recall the following facts:

(a) A subset A of a metric space is bounded iff $A \subset B(x, r)$ for some $x \in X$ and $r > 0$.

(b) A subset of a totally bounded subset is totally bounded.

Given $\varepsilon > 0$, we choose $N > \sqrt{n}/\varepsilon$. We claim that $B(0, R)$ is covered by the finite family

$$\{B(p, \varepsilon) : p = (p_1/N, \dots, p_n/N), p_i \in \mathbb{Z}, -RN \leq p_i \leq RN\}.$$

Draw pictures to convince yourself of this fact. If $x = (x_1, \dots, _n) \in B(0, R)$, then for each j there is a rational number of the form p_j/N whose distance from x_j is less than $1/N < \varepsilon/\sqrt{n}$. Then

$$d(x, (p_1/N, \dots, p_n/N)) = [\sum_j (x_j - p_j/N)^2]^{1/2} < \varepsilon.$$

This completes the proof.

Alternatively, you may prove that a cube of the form $[-R, R] \times \cdots \times [-R, R] \subset \mathbb{R}^n$ is totally bounded, by subdividing each of the sides $[-R, R]$ into N equal parts so that each subinterval $[(j-1)N, j/N]$ is of length less than δ □

Theorem 4. *Every nonempty open set $U \subset \mathbb{R}$ is the union of disjoint open intervals, at most countable in number.*

Proof. We define a relation $a \sim b$ if $[a, b]$ or $[b, a]$ is a subset of U. It is easily seen to be an equivalence relation. Therefore U is the disjoint union of the equivalence classes. We claim that any equivalence class C is an interval. For, if $x, y \in C$, then $[x, y] \subset C$. C is also open. If $x \in C$, then $x \in U$ and hence there exists an $\varepsilon > 0$ such that $(x - \varepsilon, x + \varepsilon) \subset U$. But then $(x - \varepsilon, x + \varepsilon) \in C(x) = C$. Hence U is the disjoint union of open intervals. These are at most countable. We associate a rational number $r_C \in C$ to each equivalence class C. This map is a one-one map from the set of equivalence classes into $\mathbb{Q}$ and hence the set of equivalence classes is countable. □

Lemma 5. *Let $f\colon [0, \infty) \to [0, \infty)$ be given by $f(x) = x^{1/n}$. Then f is continuous.*

Proof. Fix $x > 0$. Let $a = x^{1/n}$ and $b = y^{1/n}$. Then

$$b^n - a^n = (b - a)(b^{n-1} + \cdots + a^{n-1}.$$

Thus, we have

$$|b^n - a^n| \geq |b - a|\, nc^{n-1} \text{ if } c := \min\{a, b\}.$$

If we keep y near x so that $y > x/2$, we have

$$|y - x| \geq \left| y^{1/n} - x^{1/n} \right| n(x/2)^{(n-1)/n}.$$

Therefore,

$$\left| y^{1/n} - x^{1/n} \right| \leq \frac{1}{n} (\frac{2}{x})^{\frac{n-1}{n}} |y - x|, \quad y > x/2.$$

The continuity of f follows from this. □

Theorem 6. *Let $f\colon \mathbb{R} \to \mathbb{T} := \{z \in \mathbb{C} : |z| = 1\}$ be a continuous homomorphism. Then f is differentiable and hence $f(x) = e^{i\lambda x}$ for some $\lambda \in \mathbb{R}$.*

Proof. Since $f(0) = 1$, there exists $a > 0$ such that $b := \int_0^a f(t)\, dt \neq 0$. Now, $f(x + t) = f(x)f(t)$ for $x, t \in \mathbb{R}$. We integrate this with respect to t on $[0, a]$ and get

$$\int_0^a f(x + t)\, dt = \int_0^a f(x)f(t)\, dt = f(x) \int_0^a f(t)\, dt = bf(x). \tag{1}$$

Using a change of variable, we have

$$\int_0^a f(x + t)\, dt = \int_x^{x+a} f(u)\, du = \int_0^{x+a} f(u)\, du - \int_0^x f(u)\, du$$

is a differentiable function of x. Since $b \neq 0$, it follows from (1) that f is differentiable. Now,

$$\lim_{h \to 0} \frac{f(a + h) - f(a)}{h} = f(a) \lim_{h \to 0} \frac{f(h) - f(0)}{h} = f(a)f'(0),$$

for all $a \in \mathbb{R}$. That is, $f'(x) = f'(0)f(x)$. Hence $f(x) = e^{f'(0)x}$. Take $\lambda = f'(0)/i$. □