Monotone Functions

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1. We say that a function $f\colon J \subset \mathbb{R} \to \mathbb{R}$ is strictly increasing if for all $x, y \in J$ with $x < y$, we have $f(x) < f(y)$.

   One defines strictly decreasing in a similar way. A monotone function is either strictly increasing or strictly decreasing.

   We shall formulate and prove the results for strictly increasing functions. Analogous results for decreasing functions f can be arrived at in a similar way or by applying the result for the increasing functions to $-f$.

2. Let $J \subset \mathbb{R}$ be an interval. Let $f\colon J \to \mathbb{R}$ be continuous and 1-1. Let $a, c, b \in J$ be such that $a < c < b$. Then $f(c)$ lies between $f(a)$ and $f(b)$, that is either $f(a) < f(c) < f(b)$ or $f(a) > f(c) > f(b)$ holds.

   Proof. Since f is one-one, we assume without loss of generality that $f(a) < f(b)$. If the result is false, either $f(c) < f(a)$ or $f(c) > f(b)$.

   Let us look at the first case. Then the value $y = f(a)$ lies between the values $f(a)$ and $f(c)$ at the end points of $[a, c]$. Since $f(c) < f(a) < f(b)$, $y = f(a)$ also lies between the values of f at the end points of $[c, b]$. Hence there exists $x \in (c, b)$ such that $f(x) = y = f(a)$. Since $x > a$, this contradicts the fact that f is one-one.

   In case, you did not like the way we used y, you may proceed as follows. Fix any y such that $f(c) < y < f(a)$. By intermediate value theorem applied to the pair $(f, [a, c])$, there exists $x_1 \in (a, c)$ such that $f(x_1) = y$. Since $f(a) < f(b)$, we also have $f(c) < y < f(b)$. Hence there exists $x_2 \in (c, b)$ such that $f(x_2) = y$. Clearly $x_1 \neq x_2$.

   The second case when $f(c) > f(b)$ is similarly dealt with. □

3. **Theorem.** *Let $J \subset \mathbb{R}$ be an interval. Let $f\colon J \to \mathbb{R}$ be continuous and 1-1. Then f is monotone.*

   Proof. Fix $a, b \in J$, say with $a < b$. We assume without loss of generality that $f(a) < f(b)$. We need to show that for all $x, y \in J$ with $x < y$ we have $f(x) < f(y)$.

   (i) If $x < a$, then $x < a < b$ and hence $f(x) < f(a) < f(b)$.

   (ii) If $a < x < b$, then $f(a) < f(x) < f(b)$.

   (iii) If $b < x$, then $f(a) < f(b) < f(x)$.

$$\text{In particular, } f(x) < f(a) \text{ if } x < a \text{ and } f(x) > f(a) \text{ if } x > a. \tag{1}$$

If $x < a < y$, then $f(x) < f(a) < f(y)$ by (1).

If $x < y < a$, then $f(x) < f(a)$ by (1) and $f(x) < f(y) < f(a)$ by the last item.

If $a < x < y$, then $f(a) < f(y)$ by (1) and $f(a) < f(x) < f(y)$ by the last item.

Hence f is strictly increasing. □

4. **Proposition.** *Let J be an interval and $f\colon J \to \mathbb{R}$ be monotone. Assume that $f(J) = I$ is an interval. Then f is continuous.*

   Proof. We deal with the case when f is strictly increasing. Let $a \in J$. Assume that a is not an endpoint of J. We prove the continuity of f at a using the ε-δ definition.

   Since a is not an endpoint of J, there exists $x_1, x_2 \in J$ such that $x_1 < a < x_2$ and hence $f(x_1) < f(a) < f(x_2)$. It follows that there exists $\eta > 0$ such that $(f(a)-\eta, f(a)+\eta) \subset (f(x_1), f(x_2)) \subset I$.

   Let $\varepsilon > 0$ be given. We may assume $\varepsilon < \eta$. Let $s_1, s_2 \in J$ be such that $f(s_1) = f(a) - \varepsilon$ and $f(s_2) = f(a) + \varepsilon$. Let $\delta := \min\{a - s_1, s_2 - a\}$. If $x \in (a-\delta, a+\delta) \subset (s_1, s_2)$, then, $f(a) - \varepsilon = f(s_1) \leq f(x) < f(s_2) = f(a) + \varepsilon$, that is, if $x \in (a-\delta, a+\delta)$, then $f(x) \in (f(a)-\varepsilon, f(a)+\varepsilon)$.

   If a is an endpoint of J, an obvious modification of the proof works. □

5. Consider the n-th root function $f\colon [0,\infty) \to [0,\infty)$ given by $f(x) := x^{1/n}$. We can use the last item to conclude that f is continuous, a fact seen by us earlier.

6. Let $J \subset \mathbb{R}$ be an interval and $f\colon J \to \mathbb{R}$ be increasing. Assume that $c \in J$ is not an endpoint of J. Then
   (i) $\lim_{x\to c_-} f = \text{l.u.b. } \{f(x) : x \in J; x < c\}$.
   (ii) $\lim_{x\to c_+} f = \text{g.l.b. } \{f(x) : x \in J; x > c\}$.

7. Let the hypothesis be as in the last item Then the following are equivalent:
   (i) f is continuous at c.
   (ii) $\lim_{x\to c_-} f = f(c) = \lim_{x\to c_+} f$.
   (iii) $\text{l.u.b. } \{f(x) : x \in J; x < c\} = f(c) = \text{g.l.b. } \{f(x) : x \in J; x > c\}$.

   What is the formulation if c is an endpoint of J?

8. Let $J \subset \mathbb{R}$ be an interval and $f\colon J \to \mathbb{R}$ be increasing. Assume that $c \in J$ is not an endpoint of J. The jump at c is defined as

$$j_f(c) := \lim_{x\to c_+} f - \lim_{x\to c_-} f \equiv \text{g.l.b. } \{f(x) : x \in J; x > c\} - \text{l.u.b. } \{f(x) : x \in J; x < c\}.$$

   How is the jump $j_f(c)$ defined if c is an endpoint?

9. Let $J \subset \mathbb{R}$ be an interval and $f\colon J \to \mathbb{R}$ be increasing. Then f is continuous at $c \in J$ iff $j_f(c) = 0$. □

10. **Theorem.** *Let $J \subset \mathbb{R}$ be an interval and $f\colon J \to \mathbb{R}$ be increasing. Then the set D of points of J at which f is discontinuous is countable.*

    Proof. Assume that f is increasing. Then $c \in J$ belongs to D iff the interval $J_c := (f(c_-), f(c_+))$ is nonempty. For $c_1, c_2 \in D$, the intervals J_{c_1} and J_{c_2} are disjoint. (Why?) Thus the collection $\{J_c : c \in D\}$ is a pairwise disjoint family of open intervals. Such a collection is countable. For, choose $r_c \in J_c \cap \mathbb{Q}$. Then the map $c \mapsto r_c$ from D to $\mathbb{Q}$ is one-one. □

11. Let $f\colon \mathbb{R} \to \mathbb{R}$ be an additive homomorphism. If f is monotone, then $f(x) = f(1)x$ for all $x \in \mathbb{R}$.