Density of rational numbers using nested interval theorem

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Theorem 1 (Density of Rational Numbers in $\mathbb{R}$). *Let $x < y$ be real numbers. Then there exists a rational number $r \in \mathbb{Q}$ such that $x < y < r$.*

Proof. If x (respectively y) is a rational number, then, by Archimedean property, we can find a natural number N such that $y - x > 1/N$. Then the rational number $x + \frac{1}{N}$ (respectively $y - \frac{1}{N}$) is as required. So we assume that neither of the given numbers is rational.

Since $\mathbb{Z}$ is unbounded in $\mathbb{R}$, we can find integers m and n such that $m < x < y < n$. Consider the midpoint $(m+n)/2$. It is different from x and y. We write

$$I_0 = [m, n] = I_{01} \cup I_{02} \equiv [m, \frac{m+n}{2}] \cup [\frac{m+n}{2}, n].$$

Now, if x and y lie in different intervals, it follows that $x < \frac{m+n}{2} < y$. If they lie in the same interval, say, $[m, \frac{m+n}{2}]$, call it I_1. Now we bisect $I_1 = I_{11} \cup I_{12}$ and ask whether x and y lie in different subintervals or not. If they do, we achieved what we wanted. Otherwise, we repeat the process. We claim that at some finite stage, we must have x and y lying in different subintervals I_{n1} and I_{n2}. For, otherwise, the infinite process will give us a sequence (I_n) of nested intervals, with $\ell(I_n) = 2^{-n} n - m$ and such that $x, y \in \cap_n I_n$. Since the lengths of the intervals converge to 0, there can be only one point in $\cap I_n$. But this forces us to conclude $x = y$, a contradiction. Hence at some n-th stage, $x \in I_{n1}$ ad $y \in I_{n2}$. The common end point of these subinervals is a rational number. □

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