Nested Interval Theorem and Its Applications

S. Kumaresan
School of Math. and Stat.
University of Hyderabad
Hyderabad 500046
kumaresa@gmail.com

If $J = [a, b]$ is an interval, we let $\ell(J) := b - a$, the length of the interval. We shall repeatedly use the following two trivial observations.

Lemma 1. *If $x, y \in [a, b]$, then $|x - y| \leq b - a$.*

Proof. Without loss of generality assume that $x \leq y$ and hence $|x - y| = y - x$. Since $x \in [a, b]$, we have $a \leq x$ and hence $-x \leq -a$. Also, $y \in [a, b]$ and therefore $y \leq b$. It follows that

$$|x - y| = y - x \leq b - a.$$

□

Lemma 2. *Let $[a, b] \subset [c, d]$. Then $c \leq a \leq b \leq d$.* □

Theorem 3. [Nested Interval Theorem] *Let $J_n := [a_n, b_n]$ be intervals in $\mathbb{R}$ such that $J_{n+1} \subseteq J_n$ for all $n \in \mathbb{N}$. Then $\cap J_n \neq \emptyset$.*

If, furthermore, we assume that $\lim_{n\to\infty} \ell(J_n) \to 0$, then $\cap_n J_n$ contains precisely one point.

Proof. Note that the hypothesis means that $[a_{n+1}, b_{n+1}] \subset [a_n, b_n]$ for all n. In particular, $a_n \leq a_{n+1}$ and $b_{n+1} \leq b_n$ for all $n \in \mathbb{N}$. Observe also that if $s \geq r$, then $J_s \subset J_r$:

$$J_s \subseteq J_{s-1} \subseteq \cdots \subseteq J_{r+1} \subseteq J_r.$$

Hence, in view of Lemma 2, we have

$$a_r \leq a_s \leq b_s \leq b_r, \quad \text{in particular } a_s \leq b_r. \tag{1}$$

Let E be the set of left endpoints of J_n. Thus, $E := \{a \in \mathbb{R} : a = a_n \text{ for some } n\}$. E is nonempty.

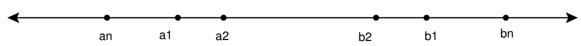

Figure 1:

We claim that b_k is an upper bound for E for each $k \in \mathbb{N}$, i.e., $a_n \leq b_k$ for all n and k. If $k \leq n$ then $[a_n, b_n] \subseteq [a_k, b_k]$ and hence $a_n \leq b_n \leq b_k$. (Draw pictures!) If $k > n$ then $a_n \leq a_k \leq b_k$. (Use Eq. 1.) Thus the claim is proved. By the LUB axiom there exists $c \in \mathbb{R}$ such that $c = \sup E$. We claim that $c \in J_n$ for all n. Since c is an upper bound for E we have $a_n \leq c$ for all n. Since each b_n is an upper bound for E and c is the least upper bound for E we see that $c \leq b_n$. Thus we conclude that $a_n \leq c \leq b_n$ or $c \in J_n$ for all n. Hence $c \in \cap J_n$.

Let us now assume further that $\ell(J_n) \to 0$ as $n \to \infty$. By the first part we know that $\cap J_n \neq \emptyset$. Let $x, y \in \cap_n J_n$. We claim that $x = y$. For, since $x, y \in J_n$ for all n, we have

$$|x - y| \leq \ell(J_n) \to 0, \text{ as } n \to \infty.$$

Thus we conclude that $|x - y| = 0$ and hence $x = y$. □

Theorem 4. *The set $\mathbb{R}$ of real numbers is uncountable.*

Proof. We shall show that the set $[0, 1]$ is uncountable. The theorem follows from this.

Assume that $[0, 1]$ is countable. That is, there exists a map $f : \mathbb{N} \to [0, 1]$ which is onto. Let $x_n := f(n) \in [0, 1]$. To arrive at a contradiction, we shall employ the Nested interval theorem. Let us subdivide the interval $[0, 1]$ into three closed subintervals of equal length. Then there exists at least one subinterval which does not contain $f(1) = x_1$. (Draw a picture. The worst possible case is when $f(1) = x_1$ happens to be either $1/3$ or $2/3$.)

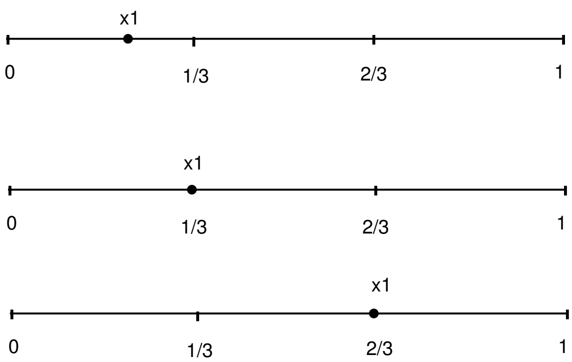

Figure 2:

Select a subinterval which does not contain x_1 and call it J_1. Let us subdivide J_1 into three equal closed subintervals. Either $x_2 = f(2) \notin J_1$ or it lies in at most two of the subintervals of J_1. In any case there exists a subinterval of J_1 which does not contain $f(2)$. Choose one such and call it J_2. We proceed along this line to construct a subinterval J_n of J_{n-1} which does not contain $f(n)$. Thus we would have obtained a nested sequence (J_n) of closed and bounded intervals. By the nested interval theorem their intersection is nonempty: $\cap_n J_n \neq \emptyset$. Let $x \in \cap J_n$. Then $x \in [0, 1]$. So there exists an $k \in \mathbb{N}$ such that $x = f(k)$. Thus $f(k) = x \in \cap_n J_n$. In particular, $f(k) \in J_k$. This contradicts our choice of J_n's. Hence our assumption that $[0, 1]$ is countable is wrong. □

Theorem 5 (Bolzano-Weierstrass)**.** *Let (x_n) be a bounded sequence of real numbers. Then there exists a convergent subsequence (x_{n_k}) of (x_n).*

Proof. Recall that a real sequence is a function $x\colon \mathbb{N} \to \mathbb{R}$ and that we let $x_n := x(n)$. Thus the image of the sequence is the subset $x(\mathbb{N}) \equiv \{x_n : n \in \mathbb{N}\} \subset \mathbb{R}$.

If the image $x(\mathbb{N})$ is finite, then there exists a real number α such that $x_n = \alpha$ for infinitely many $n \in \mathbb{N}$, say, for all $k \in S \subset \mathbb{N}$, an infinite subset. Using the well-ordering principle, we can exhibit the elements of S as an increasing sequence of natural numbers:

$$n_1 < n_2 < \cdots < n_k < \cdots$$

Then the subsequence (x_{n_k}) is the constant sequence α and hence is convergent.

So, we now assume that $x(\mathbb{N})$ is infinite. Since (x_n) is bounded, there exists a positive real number M such that

$$-M \leq x_n \leq M, \qquad \text{for all } n \in \mathbb{N}.$$

We bisect the interval $J_0 := [-M, M]$ into two subintervals of equal length, $[-M, 0]$ and $[0, M]$. Since $x(\mathbb{N}) \subset [-M, M] = [-M, 0] \cup [0, M]$, at least one of the subintervals will contain infinitely many elements of $x(\mathbb{N})$, that is, infinitely many terms of the given sequences. Call one such subinterval as J_1. Now we again bisect J_1 into two subintervals of equal length. Since $x(\mathbb{N}) \cap J_1$ is infinite by our choice of J_1, one of the subintervals of J_1 must have infinite number of elements from $x(\mathbb{N}) \cap J_1$. Call it J_2. Thus, $J_2 \subset J_1$, $\ell(J_2) = 2^{-1}\ell(J_1) = 2^{-2}\ell(J_0) = 2^{-2}2M$. Also, J_2 has infinitely many terms from (x_n).

Proceeding inductively, we construct a nested sequence of intervals J_n with the following properties:
(i) $J_{n+1} \subset J_n$ for all $n \in \mathbb{N}$.
(ii) $\ell(J_n) = 2^{-n+1}M$ for all $n \in \mathbb{N}$.
(iii) For each $n \in \mathbb{N}$, the interval J_n contains infinitely many terms of the sequence (x_n).

Using the nested interval theorem, we get a real number α such that $\alpha \in \cap_n J_n$. Now we inductively define a subsequence which will converge to α. Since J_1 contains infinitely many terms of the sequence (x_n), there exists $n_1 \in \mathbb{N}$ such that $x_1 \in J_1$. Assume that for all $1 \leq i \leq k$, we have found x_{n_i} such that $x_{n_i} \in J_i$. Now, J_{k+1} contains infinitely many terms of the given sequence and hence there exists n_k such that $n_{k+1} > n_i$ for $1 \leq i \leq k$ with the property that $x_{n_{k+1}} \in J_{k+1}$. Thus we get a subsequence (x_{n_k}).

We claim that $x_{n_k} \to \alpha$ as $k \to \infty$. For since $\alpha, x_{n_k} \in J_k$, we have

$$|x_{n_k} - \alpha| \leq \ell(J_k) = 2^{-k+1}M \to 0, \text{ as } k \to \infty.$$

This completes the proof of the theorem. □

An immediate corollary is the following:

Corollary 6. *If (x_n) is a Cauchy sequence of real numbers, then (x_n) is convergent.*

Proof. Recall the following facts.
(i) Any Cauchy sequence is bounded.
(ii) If (x_n) is a Cauchy sequence and (x_{n_k}) is a subsequence convergent to a real number α, then (x_n) converges to α.

The proof is an easy consequence of these two facts along with the last theorem. The reader should complete the proof on his own.

Let (x_n) be a Cauchy sequence of real numbers. By the first fact, (x_n) is bounded. By Bolzano-Weierstrass theorem, there exists a convergent subsequence (x_{n_k}). It follows from the second fact that the given Cauchy sequence is convergent. $\square$

Ex. 7. Let A be subset of $\mathbb{R}$. We say that a real number x is an *accumulation point* (or a cluster point) of A if for every $\varepsilon > 0$, the intersection $(x-\varepsilon, x+\varepsilon) \cap A$ contains a point other than x. (The intersection may or may not contain x!) Adapt the proof of Thm. 5 to prove the following version of Bolzano-Weierstrass theorem: If A is a bounded infinite subset of $\mathbb{R}$, there is an accumulation point of A in $\mathbb{R}$.

Theorem 8 (Intermediate Value Theorem)**.** *Let $f\colon [a,b] \to \mathbb{R}$ be continuous. Assume that $f(a)$ and $f(b)$ are of different signs, say, $f(a) < 0$ and $f(b) > 0$. Then there exists $c \in (a,b)$ such that $f(c) = 0$.*

Proof. Let $J_0 := [a,b]$. Let c_1 be the mid point of $[a,b]$. Now there are three possibilities for $f(c_1)$. It is zero, negative or positive. If $f(c_1) = 0$, then the proof is over. If not, we choose one of the intervals $[a,c_1]$ or $[c_1,b]$ so that f assumes values with opposite signs at the end points. To spell it out, if $f(c_1) < 0$, then we take the subinterval $[c_1,b]$. If $f(c_1) > 0$, then we take the subinterval $[a,c_1]$. The chosen subinterval will be called J_1 and we write it as $[a_1,b_1]$.

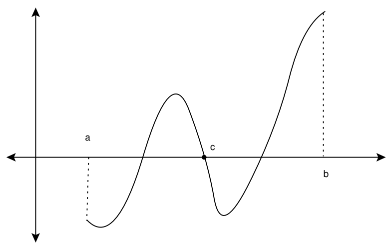

Figure 3:

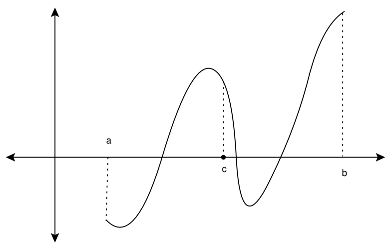

Figure 4:

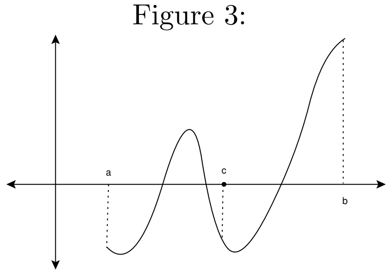

Figure 5:

We now bisect the interval J_1 and choose one of the two subintervals as $J_2 := [a_2,b_2]$ so that f takes values with opposite signs at the end points. We continue this process recursively. We thus obtain a sequence (J_n) of intervals with the following properties:
(i) If $J_n = [a_n,b_n]$, then $f(a_n) \le 0$ and $f(b_n) \ge 0$.
(ii) $J_{n+1} \subset J_n$.
(iii) $\ell(J_n) = 2^{-n}\ell(J_0) = 2^{-n}(b-a)$.

By nested interval theorem there exists a unique $c \in \cap J_n$. Since $a_n, b_n, c \in J_n$, we have

$$|c - a_n| \leq \ell(J_n) = 2^{-n}(b-a) \text{ and } |c - b_n| \leq \ell(J_n) = 2^{-n}(b-a).$$

Hence it follows that $\lim a_n = c = \lim b_n$. Since $c \in J$ and f is continuous on J, we have

$$\lim_{n\to\infty} f(a_n) = f(c) \text{ and } \lim_{n\to\infty} f(b_n) = f(c).$$

Since $f(a_n) \leq 0$ for all n, it follows that $\lim_n f(a_n) \leq 0$, that is, $f(c) \leq 0$. In an analogous way, $f(c) = \lim f(b_n) \geq 0$. We are forced to conclude that $f(c) = 0$. The proof is complete. □

Ex. 9. Deduce from Thm. 8 the standard version of the Intermediate value theorem: Let $f\colon [a,b] \to \mathbb{R}$ be continuous. Let α be a real number in between $f(a)$ and $f(b)$, that is, $f(a) < \alpha < f(b)$ or $f(b) < \alpha < f(a)$ whichever makes sense. Then there exists $c \in (a,b)$ such that $f(c) = \alpha$. *Hint:* Assume $f(a) < \alpha < f(b)$. Consider $g(x) = f(x) - \alpha$ for $x \in [a,b]$. Apply Thm. 8.

Theorem 10. *Let $f\colon [a,b] \to \mathbb{R}$ be continuous. Then f is bounded on $[a,b]$. That is, there exist $L, M \in \mathbb{R}$ such that*

$$L \leq f(x) \leq M, \text{ for all } x \in [a,b].$$

Proof. We say that a function $f\colon [a,b] \to \mathbb{R}$ is bounded above if there exists $M \in \mathbb{R}$ such that $f(x) \leq M$ for all $x \in [a,b]$. If f is not bounded above, then given any $M \in \mathbb{R}$, there exists an $x \in [a,b]$ such that $f(x) > M$.

Let $f\colon [a,b] \to \mathbb{R}$ be continuous. We prove that it is bounded above in $[a,b]$. By way of contradiction, assume that it is not bounded above. We bisect $J_0 := [a,b]$ into subintervals $[a,c]$ and $[c,b]$ of equal length as usual. Then f is not bounded above in at least one of the subintervals. For, if not, then there exist M_1 and M_2 such that $f(x) \leq M_1$ for all $x \in [a,c]$ and $f(x) \leq M_2$ for all $x \in [c,b]$. It follows that if $M = \max\{M_1, M_2\}$, then $f(x) \leq M$ for all $x \in [a,b]$, contrary to our assumption. Hence we conclude that f is unbounded in at least one of the subintervals. Select one such subinterval and call it J_1.

We again bisect J_1 and repeat the argument to conclude there exists a subinterval J_2 on which f is not bounded. Proceeding inductively, we get a sequence of intervals J_n with the following properties:

(i) $J_{n+1} \subset J_n$ for all n.
(ii) $\ell(J_n) = 2^{-n}\ell(J_0) = 2^{-n}(b-a)$.
(iii) f is unbounded on each J_n.

By nested interval theorem, there exists an $x \in \cap J_n$. Given $n \in \mathbb{N}$, by the observation made at the beginning of the proof, there exists $x_n \in J_n$ such that $f(x_n) > n$. Clearly, as seen earlier, $x_n \to x$. By continuity of f at $x \in [a,b]$, we have $\lim_n f(x_n) = f(x)$. In particular, the sequence $(f(x_n))$ is convergent. But this is impossible, since it is divergent to $+\infty$. This contradiction proves that our assumption that f is not bounded above in $[a,b]$ is wrong. Hence f is bounded above in $[a,b]$.

(Another way of arriving at a contradiction runs as follows: By continuity of f at x, for $\varepsilon = 1$, there exists a $\delta > 0$ such that for all $y \in (x-\delta, x+\delta)$, we have $|f(y) - f(x)| < 1$. It follows that for any $y \in (x-\delta, x+\delta)$, we have $|f(y)| < 1 + |f(x)|$. Since $x_n \to x$, for δ as above, we can find n_0 such that $x_n \in (x-\delta, x+\delta)$ for all $n \geq n_0$. But for such n, we have

$$n < f(x_n) < 1 + |f(x)| .$$

As a consequence, the set $\mathbb{N}$ of natural numbers is bounded above by $\max\{n_0, 1 + |f(x)|\}$. This contradicts the Archimedean property.)

In a similar way, one can prove that f is bounded below, that is, there exists $L \in \mathbb{R}$ such that $f(x) \geq L$ for all $x \in \mathbb{R}$. Or, we apply the above result to the function $g = -f$ to find a C such that $g(x) \leq C$ for all $x \in [a,b]$. This implies that $f(x) \geq -C$ for all $x \in [a,b]$. That is, f is bounded below in $[a,b]$. Thus we have shown that f is bounded above as well as below. This completes the proof of the theorem. □

Ex. 11. Let $f\colon [a,b] \to \mathbb{R}$ be a continuous functions. Show that $m := \inf\{f(x) : x \in [a.b]\}$ and $M := \sup\{f(x) : x \in [a,b]\}$ exist as real numbers and that there exist $c, d \in [a,b]$ such that $f(c) = m$ and $f(d) = M$. *Hint:* That M is finite follows from Thm. 10. If $f(x) \neq M$ for all $x \in [a,b]$, then consider $g(x) := 1/(M - f(x))$. Apply Thm. 10

Ex. 12. Use Thm. 5 to prove Thm. 10 as well as solve Exer. 11. *Hint:* Apply Thm. 5 to a sequence (x_n) in $[a,b]$ such that $f(x_n) > n$.

Theorem 13 (Heine-Borel)**.** *Let $I = [a,b]$ be a closed and bounded interval. Let $\{J_\alpha : \alpha \in \Lambda\}$ be a family of open intervals such that $J \subset \cup_\alpha J_\alpha$. Then there exists a finite subset $F \subset \Lambda$ such that $J \subset \cup_{\alpha \in F} J_\alpha$, that is, J is contained in the union of a finite number of open intervals of the given family.*

Proof. We prove this by contradiction. Assume that J is not contained in the union of **any** finite number of intervals from the given family. Bisect I as usual. Then one of the subintervals is not contained in the union of any finite numbers of J_α's. (Why?) Call one with this property as a rogue interval. Let I_1 be a rogue interval of I. If we bisect I_1 again, one of the subintervals of I_1 must be a rogue. Select one and call it I_2.

Continuing this process, we get a nested sequence (I_n) of subintervals of I such that each of them is a rogue and $\ell(I_n) \equiv 2^{-n}(b-a) \to 0$ as $n \to \infty$. By nested interval theorem, there exists a unique $x \in \cap I_n$. Now $x \in I \subset \cup J_\alpha$ so that there exists $\lambda \in \Lambda$ such that $x \in J_\lambda$. Since J_λ is an open interval, there exists $\varepsilon > 0$ such that $(x-\varepsilon, x+\varepsilon) \subset J_\lambda$. (Why? If $J_\lambda = (a_\lambda, b_\lambda)$, take $\varepsilon < \min\{x - a_\lambda, b_\lambda - x\}$.)

Choose any N such that $2^{-N}(b-a) < \varepsilon$. We claim that $I_N \subset (x-\varepsilon, x+\varepsilon)$. Let $y \in I_N$. Since $x, y \in I_N$, we have

$$|x-y| \leq \ell(I_N) = 2^{-N}(b-a) < \varepsilon.$$

Since $y \in I_N$ is arbitrary, it follows that $I_N \subset (x-\varepsilon, x+\varepsilon)$. Since $(x-\varepsilon, x+\varepsilon) \subset J_\lambda$, we deduce that $I_N \subset J_\lambda$. This is a contradiction to our choice of I_n's all of which are rogues! This contradiction proves that our assumption that I is rogue is false and hence the theorem is proved. □

Ex. 14. Deduce from Thm. 8 the standard version of the Intermediate value theorem: Let $f\colon [a,b] \to \mathbb{R}$ be continuous. Let α be a real number in between $f(a)$ and $f(b)$, that is, $f(a) < \alpha < f(b)$ or $f(b) < \alpha < f(a)$ whichever makes sense. Then there exists $c \in (a,b)$

such that $f(c) = \alpha$. *Hint:* Assume $f(a) < \alpha < f(b)$. Consider $g(x) = f(x) - \alpha$ for $x \in [a, b]$. Apply Thm. 8.

Ex. 15. Let A be subset of $\mathbb{R}$. We say that a real number x is an *accumulation point* (or a cluster point) of A if for every $\varepsilon > 0$, the intersection $(x - \varepsilon, x + \varepsilon) \cap A$ contains a point other than x. (The intersection may or may not contain x!) Adapt the proof of Thm. 5 to prove the following version of Bolzano-Weierstrass theorem: If A is a bounded infinite subset of $\mathbb{R}$, there is an accumulation point of A in $\mathbb{R}$.

Ex. 16. Let $f: [a, b] \to \mathbb{R}$ be a continuous functions. Show that $m := \inf\{f(x) : x \in [a.b]\}$ and $M := \sup\{f(x) : x \in [a, b]\}$ exist as real numbers and that there exist $c, d \in [a, b]$ such that $f(c) = m$ and $f(d) = M$. *Hint:* That M is finite follows from Thm. 8. If $f(x) \neq M$ for all $x \in [a, b]$, the consider $g(x) := 1/(M - f(x))$. Apply once again Thm. 8

Ex. 17. Use Thm. 5 to prove Thm. 8. *Hint:* Apply Thm. 5 to a sequence (x_n) in $[a, b]$ such that $f(x_n) > n$.

Theorem 18 (Density of Rational Numbers in $\mathbb{R}$)**.** *Let $x < y$ be real numbers. Then there exists a rational number $r \in \mathbb{Q}$ such that $x < y < r$.*

Proof. This proof is due to Ms. Udita, an MTTS participant.

If x (respectively y) is a rational number, then, by Archimedean property, we can find a natural number N such that $y - x > 1/N$. Then the rational number $x + \frac{1}{N}$ (respectively $y - \frac{1}{N}$) is as required. So we assume that neither of the given numbers is rational.

Since $\mathbb{Z}$ is unbounded in $\mathbb{R}$, we can find integers m and n such that $m < x < y < n$. Consider the midpoint $(m + n)/2$. It is different from x and y. We write

$$I_0 = [m, n] = I_{01} \cup I_{02} \equiv [m, \frac{m+n}{2}] \cup [\frac{m+n}{2}, n].$$

Now, if x and y lie in different intervals, it follows that $x < \frac{m+n}{2} < y$. If they lie in the same interval, say, $[m, \frac{m+n}{2}]$, call it I_1. Now we bisect $I_1 = I_{11} \cup I_{12}$ and ask whether x and y lie in different subintervals or not. If they do, we achieved what we wanted. Otherwise, we repeat the process. We claim that at some finite stage, we must have x and y lying in different subintervals I_{n1} and I_{n2}. For, otherwise, the infinite process will give us a sequence (I_n) of nested intervals, with $\ell(I_n) = 2^{-n}n - m$ and such that $x, y \in \cap_n I_n$. Since the lengths of the intervals converge to 0, there can be only one point in $\cap I_n$. But this forces us to conclude $x = y$, a contradiction. Hence at some n-th stage, $x \in I_{n1}$ ad $y \in I_{n2}$. The common end point of these subintervals is a rational number. $\square$

Volterra's Proof of Nonexistence of a Function

Consider the function defined on $(0, 1)$ by

$$f(x) = \begin{cases} 1/q, & \text{if } x = \frac{p}{q} \text{ with g.c.d}(p, q) = 1, \\ 0 & \text{if } x \text{ is irrational.} \end{cases}$$

It is easy to show that f is continuous at each irrational point and discontinuous at all rational points of $(0, 1)$. One may now want know whether there exists a function on $(0, 1)$ which is continuous at all rationals and discontinuous at all irrationals. The nonexistence of such a function is usually proved using Baire Category theorem. Volterra proved this using a very ingenious idea without using Baire's theorem. His proof uses the nested interval theorem, density of rationals and irrationals and the existence of a function which is discontinuous only at rationals! We shall indicate his proof below.

Let us assume that there exists $g\colon (0,1) \to \mathbb{R}$ which is continuous at the rational points and discontinuous at irrationals. Let f be the function defined above. Choose any irrational point $x_0 \in (0,1)$. By continuity of f at x_0, given $\varepsilon = 1/2$, there exists a $\delta > 0$ such that

$$|f(x) - f(x_0)| < 1/2, \text{ whenever } |x - x_0| < \delta.$$

Select points $a_1 < b_1 \in (x_0 - \delta, x_0 + \delta)$. Then for all $x, y \in [a_1, b_1]$, we have

$$|f(x) - f(y)| \leq |f(x) - f(x_0)| + |f(x_0) - f(y)| < 1/2 + 1/2 = 1.$$

We now select a rational point y_0 in the open interval (a_1, b_1). We repeat the above argument using now the continuity of g at y_0 to construct a closed interval $[c_1, d_1] \subset (a_1, b_1)$ such that

$$|g(x) - g(y)| < 1, \text{ for all } x, y \in [c_1, d_1].$$

Note that we have

$$|f(x) - f(y)| < 1 \text{ and } |g(x) - g(y)| < 1 \text{ for all } x, y \in [c_1, d_1].$$

We repeat this argument replacing the open interval $(0,1)$ by the open interval (c_1, d_1) to find a closed interval $[c_2, d_2] \subset (c_1, d_1)$ such that

$$|f(x) - f(y)| < 1/2 \text{ and } |g(x) - g(y)| < 1/2 \text{ for all } x, y \in [c_2, d_2].$$

By induction we construct a sequence of nested intervals $[c_k, d_k] \subset (c_{k-1}, d_{k-1})$ for $k \in \mathbb{N}$ with the property that

$$|f(x) - f(y)| < 2^{-k+1} \text{ and } |g(x) - g(y)| < 2^{-k+1} \text{ for all } x, y \in [c_k, d_k].$$

By the nested interval theorem, there exists a unique point $a \in [c_k, d_k]$ for $k \in \mathbb{N}$. Note that by the fact that $[c_{k+1}, d_{k+1}] \subset (c_k, d_k)$, the point $a \in (c_k, d_k)$ for all k. We now show that f and g are continuous at a. Given $\varepsilon > 0$, choose n such that $2^{-n} < \varepsilon$. Then, for $x \in [c_{n+1}, d_{n+1}]$, we have

$$|f(x) - f(a)| < \varepsilon \text{ and } |g(x) - g(a)| < \varepsilon. \tag{2}$$

Since $a \in (c_{n+1}, d_{n+1})$, we can find $\delta > 0$ such that $(a - \delta, a + \delta) \subset (c_{n+1}, d_{n+1})$. It is clear that if $|x - a| < \delta$, then (2) holds, that is, f and g are continuous at a. It follows then that a must be rational as well as irrational, which is absurd! Thus we conclude no such f exists.

Remark 19. I learnt this proof from Dr. V. Sholapurkar, S.P. College, Pune.