Characterization of Compact Metrizable Spaces via Metrics and Continuous Functions

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A topological space X is said to be *metrizable* if there exists a metric on X such that the given topology coincides with the topology defined by the metric.

Theorem 1. *For a metrizable topological space X, the following properties are equivalent:*
1. *X is compact.*
2. *Every metric on X inducing the given topology is bounded.*
3. *Every continuous (real valued) function on X is bounded.*

Proof. We now prove the theorem according to the pattern:

$$(1) \implies (2) \implies (3) \implies (1).$$

(1) $\implies$ (2): If X is compact, so is $X \times X$. Any metric d on X inducing the given topology on X is a continuous function on $X \times X$, whence bounded.

(2) $\implies$ (3): Let f be a continuous function on X. We then push points of X apart at distances bounded from below by f using the following standard technique. Consider the *graph* Z of f:

$$Z := \{(x, f(x)) : x \in X\} \subseteq X \times \mathbb{R}.$$

The map $i\colon X \hookrightarrow Z$ given by $x \mapsto (x, f(x))$ is then a homeomorphism of X onto Z, its inverse being given by the restriction of the first projection $p\colon X \times \mathbb{R} \longrightarrow X$ to Z. The space $X \times \mathbb{R}$ with the product topology is metrizable; e.g. one may take the metric δ defined as

$$\delta((x, s), (y, t)) := d(x, y) + |t - s|.$$

Pulling this metric back to X using the map i therefore equips X with a metric d' inducing the topology given by d, which therefore by assumption is bounded, by a constant $B > 0$, say. Now by construction

$$d'(x, y) = d(x, y) + |f(y) - f(x)|,$$

so d' being bounded by B implies

$$|f(y)| \leq |f(x)| + B,$$

for all $x, y \in X$. If we fix $x \in X$, the inequality above shows that f is bounded.

(3) $\implies$ (1): We show that on any noncompact metrizable space X there exists a continuous unbounded function. Let d be any metric on X inducing the given topology and let X' be the completion of X with respect to d. We distinguish the cases X' being compact and being not so.

Case (i): X' compact. Since X is assumed to be noncompact, $X \neq X'$ whence $X' \setminus X$ is not empty. Let x_∞ be a point in $X' \setminus X$. Since X is dense in X', the function f defined by $f(x) := 1/d(x, x_\infty)$ is then a continuous function on X which is not bounded.

Case (ii): X' noncompact. If f is a continuous unbounded function on X', its restriction to X is a continuous unbounded function on X. So we may assume X itself is complete. According to the standard characterization of compactness, X cannot be totally bounded since it is assumed to be noncompact. So there is a real number $\varepsilon > 0$ such that X cannot be covered by finitely many closed ε-balls. Let x_1 be any point in X and put $r_1 := \varepsilon$. Then the closed ball $B[x_1, r_1]$ does not cover X. So there is x_2 in $X \setminus B[x_1, r_1]$. The latter complement being open there is r_2 with $r_1 \geq r_2 > 0$ such that $B[x_2, r_2] \subseteq X \setminus B[x_1, r_1]$. The balls $B[x_1, r_1]$ and $B[x_2, r_2]$ together do not cover X, so there are x_3 and r_3 with $B[x_3, r_3] \subseteq X \setminus B[x_1, r_1] \cup B[x_2, r_2]$ and $r_1 \geq r_2 \geq r_3 > 0$. Continuing this way we obtain sequences $x_1, x_2, \ldots$ and $r_1 \geq r_2 \geq \cdots > 0$. They have the property that the balls $B[x_k, r_k]$ are mutually disjoint. Now we define $f\colon X \to \mathbb{R}$ as follows:

$$f(x) := \sum_{k=1}^{\infty} k \cdot \frac{d(x, X \setminus B(x_k, r_k))}{d(x, x_k) + d(x, X \setminus B(x_k, r_k))}\,.$$

(Visualize f in the case of $X = \mathbb{R}$ and $x_k = k$ and $r_k = 2^{-k}$, say.) If the k-th term of the sum that defines $f(z)$ is nonzero, it means that $z \in B[x_k, r_k]$ and hence all other terms of the series that defines $f(z)$ are zero, since the balls $B[x_k, r_k]$ are mutually disjoint. Hence the series is convergent and $f(x)$ makes sense for any $x \in X$. We thus get a well-defined function f on X. Since $f(x_k) = k$, f is not bounded. It is easily seen to be continuous on X. For, if $x \in U := X \setminus \cup_{k \in \mathbb{N}} B[x_k, r_k]$, then $f(x) = 0$ and since U is open (why?), f is zero in an open set containing x. If $x \in B[x_k, r_k]$, then f is just the k-th term of the series, which is continuous. This finishes the proof. □

Remark 2. The case (ii) of (3) $\implies$ (1) can also be seen as follows. If X is not totally bounded, there is some $\varepsilon > 0$ such that no finite collection of balls of radius ε covers X. So we can pick x_1 in X, $x_2 \in X \setminus B(x_1, \varepsilon)$, $x_3 \in X \setminus (B(x_1, \varepsilon) \cup B(x_2, \varepsilon))$, and so on. Each $d(x_i, x_j) \geq \varepsilon$ for $i \neq j$. Hence there are no nonconstant Cauchy sequences among the x_i. So, the set $\{x_i\}$ is closed in X and also discrete. If we now define $f(x_i) = i$, then f is continuous function on the discrete set. We can extend this by the Tietze theorem to $f\colon X \longrightarrow \mathbb{R}$. The function f is clearly an unbounded continuous function.