Additive Homomorphisms of $\mathbb{R}$ and Nonmeasurable Sets

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This article arose out of two standard exercises in a Course in Real Analysis and a third exercise bordering on algebra and set theory. We shall state them below, discuss them in detail later and show how to deduce the existence of a non-measurable set using few more standard exercises in an M.Sc. course.

Ex. 1. Let $f: \mathbb{R} \to \mathbb{R}$ be a continuous additive homomorphism, that is, $f(x+y) = f(x)+f(y)$ for all $x, y \in \mathbb{R}$. Then $f(x) = f(1)x$ for all $x \in \mathbb{R}$. In particular, f is a linear map from $\mathbb{R}$ to $\mathbb{R}$.

Ex. 2. Let $f: \mathbb{R} \to \mathbb{R}$ be a measurable homomorphism. Then $f(x) = f(1)x$ for all $x \in \mathbb{R}$.

Ex. 3. There exists additive homomorphisms $f: \mathbb{R} \to \mathbb{R}$ which are not linear *over* $\mathbb{R}$. Such homomorphisms are necessarily discontinuous.

The additive homomorphisms which are not linear over $\mathbb{R}$ are necessarily discontinuous, by Ex. 1. In fact, they are not even measurable by Ex. 2. This prompts us to ask: Can one use them to prove the existence of nonmeasurable sets? We answer this affirmatively at the end of this article.

A typical solution of Ex. 1 runs as follows. Since $f(x + y) = f(x) + f(y)$ for all $x, y \in \mathbb{R}$, we have $f(2x) = f(x+x) = f(x)+f(x) = 2f(x)$. It follows by induction that $f(nx) = nf(x)$ for all $x \in \mathbb{R}$ and $n \in \mathbb{N}$. Since f is a group homomorphism, it follows that $f(-x) = -f(x)$ for all $x \in \mathbb{R}$ and hence $f(nx) = nf(x)$ for all $x \in \mathbb{R}$ and $n \in \mathbb{Z}$. Since $1 = n \cdot (1/n)$, we see that

$$f(1) = f(n\frac{1}{n}) = nf(1/n), \text{ so that } f(1/n) = \frac{1}{n}f(1).$$

Hence for any rational number $r = m/n$, we deduce that $f(m/n) = f(m\frac{1}{n}) = mf(1/n) = \frac{m}{n}f(1)$. We have so far shown that $f(r) = rf(1)$ for any $r \in \mathbb{Q}$. Now, we consider the continuous function g defined by $g(x) = f(1)x$. Then the two continuous functions f and g agree on $\mathbb{Q}$: $f(r) = g(r)$ for $r \in \mathbb{Q}$. If $x \in \mathbb{R}$ is given, by the density of rationals in $\mathbb{R}$, we can find a sequence (r_n) in $\mathbb{Q}$ such that $\lim r_n = x$. Since f and g are continuous at x and since $f(r_n) = g(r_n)$ for all n, we see that

$$f(x) = \lim f(r_n) = \lim g(r_n) = g(x).$$

Since $x \in \mathbb{R}$ was arbitrary, we have shown that $f(x) = g(x) = xf(1)$ for all $x \in \mathbb{R}$.

There is an extension of this result which is not so well-known.

Lemma 4. *Let $f\colon \mathbb{R} \to \mathbb{R}$ be an additive (not necessarily continuous) homomorphism. Assume that there exists a non-degenerate interval J and a constant $M > 0$ such that $|f(x)| \le M$ is for all $x \in J$. Then f is continuous and hence is of the form $f(x) = xf(1)$ for $x \in \mathbb{R}$.*

Proof. Let the end points of J be $a < b$. Then $a - b < 0 < b - a$. We observe that the set $J - J := \{x - y : x, y \in J\}$ contains the open interval $(a - b, b - a)$. Choose a $\delta > 0$ so that $(-\delta, \delta) \subset (a - b, b - a) \subset J - J$.

We show that f is continuous at 0. Let $\varepsilon > 0$ be given. We claim that $|f(t)| \le 2M$ for all $t \in (-\delta, \delta)$. For, any such t is of the form $t = x - y$ with $x, y \in J$ and hence,

$$|f(t)| = |f(x - y)| = |f(x) - f(y)| \le |f(x)| + |f(y)| \le 2M.$$

Now choose $N \in \mathbb{N}$ such that $2M/N < \varepsilon$. Then for t with $|t| < \delta/N$, we have

$$|f(t) - f(0)| = |f(t)| = |f(Nt/N)| = \left|\frac{1}{N} f(Nt)\right| = \frac{1}{N}|f(Nt)| \le \frac{2M}{N} < \varepsilon. \tag{1}$$

Thus f is continuous at 0.

The continuity of f at other points follows easily now. Let $x \in \mathbb{R}$ and $\varepsilon > 0$ be given. Since f is continuous at 0, for the given $\varepsilon > 0$, there exists a $\delta > 0$ such that $|f(t)| < \varepsilon$ for t with $|t| < \delta$. Let $y \in \mathbb{R}$ be such that $|y - x| < \delta$. We then have

$$|f(y) - f(x)| = |f(y - x)| < \varepsilon, \text{ since } |y - x| < \delta.$$

Since f is now a continuous homomorphism from $\mathbb{R}$ to $\mathbb{R}$, the result follows from Ex. 1. □

Corollary 5. . *Let $f\colon \mathbb{R} \to \mathbb{R}$ be an additive homomorphism which is bounded on a set E. Assume that $E - E$ contains a neighbourhood of 0. Then f is continuous and hence $f(x) = xf(1)$ for all $x \in \mathbb{R}$.*

Proof. If you go through the proof of Lemma 4, you will realize that we needed only the following fact about J. The set $J - J$ contains an open interval around 0. □

Ex. 6. Let $f\colon \mathbb{R}^n \to \mathbb{R}^m$ be an additive homomorphism. Assume that f is continuous at 0. Show that f is continuous on $\mathbb{R}^n$.

More generally, let G be a group and d be a metric on G which is translation invariant: $d(ax, ay) = d(x, y)$ for all $a, x, y \in G$. Let G be given the metric topology. Let H be another such entity. Let $f\colon G \to H$ be a group homomorphism which is continuous at the identity of G. Show that f is continuous on G. (This result extends to all 'topological groups'.)

Proposition 7. *Let $f\colon \mathbb{R} \to \mathbb{R}$ be measurable and an additive homomorphism. Then $f(x) = xf(1)$ for each $x \in \mathbb{R}$.*

Proof. We show that Corollary 5 can be applied to any f satisfying the hypothesis of the Proposition.

Let $E_n := \{x \in \mathbb{R} : |f(x)| \le n\}$ for $n \in \mathbb{N}$. Clearly, each E_n is measurable and we have $\mathbb{R} = \cup_n E_n$. Hence at least one of the E_n's must have positive measure. For, otherwise, $\mathbb{R}$ is a countable union of sets of measure zero and hence $\mathbb{R}$ itself is of measure zero, a contradiction. Let N be such that $E := E_N$ is of positive measure. It is well-known that $E - E$ has a neighbourhood of 0. (See Lemma 9.) Since $|f|$ is bounded by N on E, the result follows. □

Remark 8. With the above notation, we observe that $E_k = kE_1 := \{kx : x \in E_1\}$. For, if $y \in E_k$, and if we set $x = y/k$, then $|f(x)| = |f(y/k)| = |f(y)|\, k \leq 1$ etc. Thus each E_k is measurable and we have $m(E_k) = km(E_1)$. Hence all of them will have positive measure! We shall need the argument in the sequel.

Lemma 9. *Let $E \subset \mathbb{R}$ be measurable. Assume that $m(E) > 0$. Then there exists $\delta > 0$ such that*

$$(-\delta, \delta) \subset E - E := \{x - y : x, y \in E\}.$$

Proof. One makes decisive use of the regularity of the Lebesgue measure: If E is any set of positive finite measure, then we have

$$m(E) = \sup_{K \text{ compact}, \, K \subset E} m(K) = \inf_{G \text{ open}, \, E \subset G} m(G).$$

We first prove the lemma assuming the given set E is a compact set, say K. Note that $m(K) < \infty$. By regularity, we can find an open set $G \supseteq K$ such that $m(G) < 2m(K)$. Since K is compact, $\mathbb{R} \setminus G$ is closed and they are disjoint, it follows that $\delta := d(K, \mathbb{R} \setminus G) > 0$. We plan to show that $(-\delta, \delta) \subset K - K$.

Let $x \in \mathbb{R}$ be such that $|x| < \delta$. Then the set $x + K$ is measurable with $m(x + K) = m(K)$. We make two observations: (i) $x + K \subset G$ and (ii) $x + K$ and K are not disjoint.

If (i) were false, then there exists $y = x + k \in x + K$ such that $x \notin G$. But then, $d(k, y) = |x + k - k| = |x| < \delta$ so that $d(K, G^c) < \delta$, a contradiction. This shows that (i) holds.

If (ii) were false, then $x + K \subset G$ and $K \subset G$ are disjoint measurable subsets so that

$$m(G) \geq m(K \cup (x + K)) = m(x + K) + m(K) = 2m(K),$$

a contradiction. It follows that there exists $z \in (x + K) \cap K$. If we write $z = x + k_1 = k_2$, then $x = k_2 - k_1$. That is, $x \in K - K$. Since $x \in (-\delta, \delta)$ is arbitrary, we conclude that $(-\delta, \delta) \subset K - K$.

Returning to the general case, let E be measurable with $m(E) > 0$. Since $E = \cup_{k \in \mathbb{N}}(E \cap (-k, k))$, at least one of $E \cap (-k, k)$ is of finite positive measure. Replacing E with $E \cap (-k, k)$, we may assume that $0 < m(E) < \infty$. Using the regularity of the Lebesgue measure, we can find a compact $K \subset E$ with $m(K) > 0$. By the first part, there exists $\delta > 0$ such that $(-\delta, \delta) \subset K - K \subset E - E$. □

Lemma 10. *There exists homomorphisms $f \colon \mathbb{R} \to \mathbb{R}$ which are not ($\mathbb{R}$)-linear, that is which are not of the form $f(x) = ax$ for some $a \in \mathbb{R}$ and for all $x \in \mathbb{R}$.*

Proof. This is based on a simple but useful trick. (See also Ex. 11 below.) We consider $\mathbb{R}$ as a vector space over the field $\mathbb{Q}$. Then an application of Zorn's lemma establishes the existence of a basis $\{e_i : i \in I\}$ of the vector space $\mathbb{R}$ over $\mathbb{Q}$. (This basis is called by analysts a Hamel basis of $\mathbb{R}$.) Note that the cardinality of the set I is same as that of $\mathbb{R}$. Now, any $x \in \mathbb{R}$ can be written uniquely as $x = \sum_{i \in I} x_i e_i$ where the real numbers $x_i \neq 0$ only for finitely many i. Let $\sigma\colon I \to I$ be a nontrivial permutation (bijection). We then define $f_\sigma(x) := f_\sigma(\sum_i x_i e_i) = \sum_i x_{\sigma(i)} e_i$. This map is a $\mathbb{Q}$-linear map from the $\mathbb{Q}$-vector space $\mathbb{R}$ to itself and in particular an additive homomorphism.

If σ and τ are permutations of I, then $f_\sigma = f_\tau$ iff $\sigma = \tau$. Hence the cardinality of the set of all such f_σ is the same as that of the set of permutations of I. How many permutations of I are there? The set of permutations of I have the same cardinality as that of I^I, which is $\mathbb{R}^\mathbb{R}$. But the set of functions of the form $f(x) = ax$ have the cardinality $|\mathbb{R}|$. Since $\left|\mathbb{R}^\mathbb{R}\right| > |\mathbb{R}|$, the result follows. □

Ex. 11. Let G be a (possibly infinite) group with at least four elements, such that $g^2 = e$ for all $g \in G$. Show that G has a nontrivial automorphism. *Hint:* Think of G as a vector space over the field $\mathbb{Z}_2$.

Now we are ready to prove the existence of a non-measurable set. Consider an additive homomorphism $f\colon \mathbb{R} \to \mathbb{R}$ which is not linear. Consider $E_k := \{x \in \mathbb{R} : |f(x)| \leq k\}$. The we claim that each E_k is nonmeasurable! As observed in Remark 8, we have $E_k = kE_1$ and hence E_1 is measurable iff E_k is measurable for each k. If E_1 is measurable, then either $m(E_1) = 0$ or $m(E_1) > 0$. If the first case holds true, then $m(E_k) = km(E_1) = 0$ so that $\mathbb{R} = \cup_k E_k$ is of measure zero, a contradiction. If, on the other hand, if $m(E_1) > 0$, then by Lemma 9, $E - E$ contains a neighbourhood of zero. It follows from Cor. 5 that f is linear, a contradiction. Thus we are forced to conclude that E_k is not measurable for any k.

Remark 12. Note that an additive homomorphism $f\colon \mathbb{R} \to \mathbb{R}$ which is not linear over $\mathbb{R}$ is nonmeasurable, either because $E_1 := f^{-1}([-1, 1])$ is not measurable or because of Prop. 8. However, note that the fact that f is non-measurable would not have helped us in concluding that E_1 is not measurable.