Nonorientable Manifolds

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Abstract

We give two examples of nonorientable manifolds. Most often no rigorous proof is given to establish the nonorientability in text-books/lectures. Here are the details! Please go through the items 1-3 and verify them as suggested. As I typed this in a hurry, there could be typing/mathematical mistakes. So go through this with a keen eye for details..

Mobius band

There is a neat geometric realization of Mobius band. It is described as the surface of revolution got by revolving the line segment $(-1,1)e_3$ around the circle c of radius 2 with center at the origin in the xy-plane, but moving the line segment along the circle in such a way that if we have moved a distance u-radians, the angle between the line segment at that point and e_3 is $u/2$ and the line segment at u lies in the plane perpendicular to the circle, that is, perpendicular to $c'(u)$. More explicitly, we have the following parameterization:

$$\varphi(u,v) = 2(\cos u, \sin u, 0) + v(\cos u \sin(u/2), \sin u \sin(u/2), \cos(u/2)).$$

An atlas is given as follows:

$$\begin{aligned}
\varphi_1(u,v) &= ((2+v\sin(u/2))\cos u, (2+v\sin(u/2))\sin u, v\cos(u/2)), \\
&\qquad\qquad \text{for } (u,v) \in V_1 := (0,2\pi)\times(-1,1) \\
\varphi_2(u,v) &= ((2+v\sin(u/2))\cos u, (2+v\sin(u/2))\sin u, v\cos(u/2)), \\
&\qquad\qquad \text{for } (u,v) \in V_2 := (-\pi/2,\pi/2)\times(-1,1).
\end{aligned}$$

The following are easily checked:

1. $\varphi_1 = \varphi_2$ on $(0,\pi/2)\times(-1,1)$.
2. Let $W_1 := (3\pi/2, 2\pi)\times(-1,1)$ and $W_2 := (-\pi/2, 0)\times(-1,1)$. Then $\varphi_1(W_1) = \varphi_2(W_2)$. The transition map $\varphi_2^{-1}\circ\varphi_1$ maps W_1 homeomorphically onto W_2: **Verify these claims!**

$$\varphi_2^{-1}\circ\varphi_1(u,v) = (u-2\pi, -v), \text{ for } (u,v)\in W_1.$$

3. Thus the jacobian of the transition map on the open set $Z_1\cup Z_2 := (0,\pi/2)\times(-1,1)\cup(3\pi/2,2\pi)\times(-1,1)$ takes the value 1 on Z_1 and -1 on Z_2.

From these observations, it follows that the Mobius band is not orientable. The reasoning is as follows. If there exists a nowhere vanishing 2-form ω on the Mobius band, then we can write $\omega = f(u,v)du \wedge dv$ on U_1 with $f \neq 0$. Since U_1 is connected either $f > 0$ on U_1 or $f < 0$ on U_1. Assume that $f > 0$ on U_1. Similarly, if $\omega = g du_1 \wedge du_2$ on U_2, either $g > 0$ or $g < 0$ on U_2. But this is not possible as there exist points $p, q \in U_1 \cap U_2$ at which the jacobian the transition map takes opposite signs, say positive at p and negative at q. So, if we express ω at these points in terms of, say, u, v, then

$$\begin{aligned}\omega_p &= f(p)du \wedge dv = g(p)du_1 \wedge du_2 = g(p)\det(\partial(u_1,v_1)/\partial(u,v))du \wedge dv \\ \omega_q &= f(q)du \wedge dv = g(q)du_1 \wedge du_2 = g(q)\det(\partial(u_1,v_1)/\partial(u,v))du \wedge dv.\end{aligned}$$

If $g > 0$ on U_2, the above equations say that $f(p) > 0$ while $f(q) < 0$, a contradiction. If $g < 0$ on U_2, a similar argument applies.

One can also establish the non-orientability of the Mobius band M assuming the existence of an oriented atlas. Let $\{(V_i, \psi_i, U_i)\}$ be an oriented atlas of M. Let $U_1^{\pm}$ be the set of points $p = (u,v) \in U_1$ with the following property:

> There exists an i such that $p \in U_i \cap U_1$ and the determinant of $D(\psi_i^{-1} \circ \varphi_1)$ is positive (respectively negative) at p.

The following are easy to verify:

1. $U_i^{\pm}$ is an open set.
2. $U_i^+ \cap U_i^- = \emptyset$. For, if $p \in U_i^+ \cap U_i^-$, then there exist i, j such that $p \in U_1 \cap U_i \cap U_j$, we have $\det(D(\psi_i^{-1} \circ \varphi_1)(p)) > 0$ and $\det(D(\varphi_j^{-1} \circ \varphi_1)(p)) < 0$. As $\psi_i^{-1} \circ \varphi_j = (\varphi_1^{-1} \circ \varphi_1) \circ (\varphi_j^{-1} \circ \varphi_1)^{-1}$, it follows that $\det(\psi_i^{-1} \circ \varphi_j)(p) < 0$. This contradicts the assumption that they are charts from an oriented atlas.
3. $U = U_1^+ \cup U_1^-$. As $\det(D(\psi_i^{-1} \circ \varphi_1)(p)) \neq 0$, p has to lie in one of the sets.

These observations lead us to conclude that either $U_1 = U_1^+$ or $U_1 = U_1^-$. Assume $U_1 = U_1^+$. By a similar reasoning, we conclude that $U_2 = U_2^+$ or $U_2 = U_2^-$. Assume for definiteness sake $U_2 = U_2^+$.

Choose $q \in U_1 \cap U_2$ such that $\det(D(\varphi_2^{-1} \circ \varphi_1))(q) < 0$. We choose i, j be such that

$$\det(D(\psi_i^{-1} \circ \varphi_1))(q) > 0 \text{ and } \det(D(\psi_j^{-1} \circ \varphi_2))(q) < 0.$$

Observe that

$$\psi_i^{-1} \circ \psi_j = (\psi_i^{-1} \circ \varphi_1) \circ (\varphi_1^{-1} \circ \varphi_2) \circ (\psi_j^{-1} \circ \varphi_2)^{-1}.$$

This leads us to conclude that $\det(D(\psi_i^{-1} \circ \psi_j))(q) < 0$, a contradiction. $\square$

Recall that the Mobius band M can be obtained as a quotient space of a rectangular strip with horizontal sides removed. The points on the vertical sides are identified in the opposite direction, as indicated in the picture. Then M has two coordinate neighborhoods U_1, the open rectangular strip (that is, without the vertical sides) and U_2, the dotted portion on the left side of Figure 1 on page 3, (identified with the dotted one on the right hand side as indicated) along with the open set in M to the right of the line $x = \frac{1}{2}$.

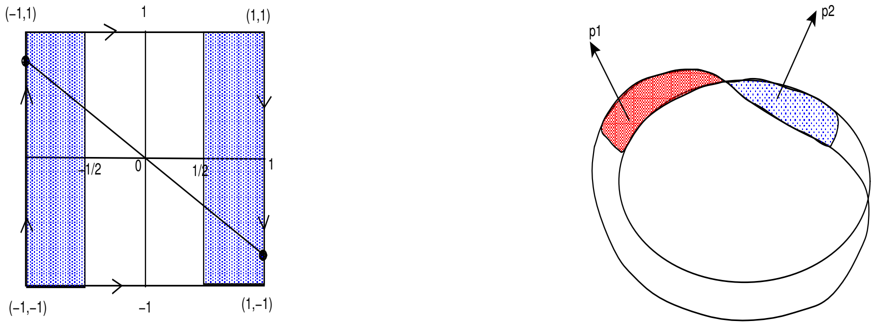

Figure 1: Möbius band

In symbols, we have

$$U_1 = \{(x, y) : x \neq \pm 1\},$$
$$U_2 = \Big\{(x, y) : \frac{1}{2} < x \leq 1 \text{ and } -1 \leq x < -\frac{1}{2}\Big\}.$$

The coordinates are given by φ_1, identity on U_1 and φ_2 on U_2 by

$$\varphi_2(x, y) = \begin{cases} (x, y) & \text{if } \frac{1}{2} < x \leq 1; \\ (2 + x, -y) & \text{if } -1 \leq x < -\frac{1}{2}. \end{cases}$$

Note that as $(1, y) = (-1, -y)$ in M, we need to check that φ_2 is well defined. It is so and it is smooth. Clearly we have on

$$U_1 \cap U_2 = \{(x, y) : \frac{1}{2} < x < 1\} \cup \{(x, y) : -1 < x < -1/2\}.$$

Note that U_2 is *twisted* and then joined to U_1. (See Figure 1.) The Jacobian of the coordinate changes has determinant $+1$ on the first set on the right side above and determinant -1 on the second set. Now , as earlier, this implies that M is *not* orientable.

Even dimensional Projective spaces

Another example of nonorientable manifolds is the even-dimensional real projective spaces. The underlying set is the set $\mathbb{P}^n(\mathbb{R})$ of lines in $\mathbb{R}^{n+1}$ passing through the origin. There is another way of looking at $\mathbb{P}^n(\mathbb{R})$. On $\mathbb{R}^{n+1} \setminus \{0\}$ we introduce the equivalence relation $\sim$ defined as follows:

$$x \sim y \iff \lambda x = y \text{ with } \lambda \in \mathbb{R},\ \lambda \neq 0.$$

An equivalence class $[x]$ containing $x \in \mathbb{R}^{n+1}$ can be identified with the line through the origin in $\mathbb{R}^{n+1}$ joining any point in the equivalence class. This second way of looking at $\mathbb{P}^n(\mathbb{R})$ is what we are going to exploit to endow $\mathbb{P}^n(\mathbb{R})$ with a manifold structure. In $\mathbb{P}^n(\mathbb{R})$ we have some very special sets U_i for $1 \leq i \leq n+1$ defined by

$$U_i := \{[x] \in \mathbb{P}^n(\mathbb{R}) : x_i \neq 0\}.$$

Notice that the definition of U_i is independent of the choice of the representative of $[x]$. For, if $y \in [x]$, then $y_j = \lambda x_j$ for all $1 \leq j \leq n+1$ and for some nonzero real number λ. On these sets we define ψ_i as follows:

$$\psi_i([y]) := \left(\frac{x_1}{x_i}, \ldots, \frac{x_{i-1}}{x_i}, \frac{x_{i+1}}{x_i}, \ldots, \frac{x_{n+1}}{x_i}\right) \in \mathbb{R}^n$$

for any $x = (x_1, \ldots, x_{n+1})$ in the equivalence class $[y]$. Again notice that the map ψ_i is well defined, that is, it is independent of the choice of $x \in [y]$.

It is now an easy exercise to check that the family $\{(U_i, \varphi_i)\}$ satisfies the assumptions of the theorem. Thus $\mathbb{P}^n(\mathbb{R})$ is a manifold of dimension n. If you know about quotient topology you may be interested in verifying that the topology we gave to $\mathbb{P}^n(\mathbb{R})$ is the quotient topology induced from $\mathbb{R}^{n+1} \setminus \{0\}$ with respect to $\sim$.

Let $\pi\colon S^n \to \mathbb{P}^n(\mathbb{R})$ be the quotient map. Let ω be the n-form on S^n given by

$$\omega = (-1)^{i-1}\frac{1}{x_i} dx_1 \wedge \cdots \wedge dx_{i-1} \wedge dx_{i+1} \cdots dx_{n+1},$$

on the set $U_i := \{p \in S^n : x_i(p) \neq 0\}$. Let $\sigma\colon S^n \to S^n$ be the diffeomorphism $\sigma(x) = -x$. Then

$$\sigma^*(\omega) = (-1)^{i-1}\frac{1}{-x_i} d(-x_1) \wedge \cdots \wedge d(-x_{i-1}) \wedge d(-x_{i+1} \cdots d(-x_{n+1}) = (-1)^{n+1}\omega.$$

If α is a nowhere vanishing n-form on $\mathbb{P}^n(\mathbb{R})$, then $\pi^*(\alpha) = f\omega$ fo r some nowhere vanishing smooth function $f \in C^\infty(S^n)$. Since $\pi \circ \sigma = \pi$, we see that

$$f\omega = \pi^*(\alpha) = \sigma^*\pi^*(\alpha) = (f \circ \sigma)(-1)^{n+1}\omega.$$

Hence if n is even, $f \circ \sigma = -f$. This means that if $f(p) > 0$, then $f(-p) < 0$. Since f is nonzero and S^n is connected this leads to a contradiction.