Burnside Theorem on Matrix Algebras and Its Application to Group Theory

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In 1905, Burnside proved the following remarkable result on groups of invertible matrices over the complex field $\mathbb{C}$:

Theorem 1 (Burnside)**.** *Let G be a group of invertible $n \times n$ matrices over $\mathbb{C}$. Then G has no nontrivial invariant subspaces in $\mathbb{C}^n$ iff G contains n^2 linearly independent matrices, that is, iff the $\mathbb{C}$-span of G in $\mathbf{M}_n(\mathbb{C})$ is $\mathbf{M}_n(\mathbb{C})$ itself.*

Remark 2. The "if" part is easy, since $\mathbf{M}_n(\mathbb{C})$ has no nontrivial invariant subspaces in $\mathbb{C}^n$ (the "trivial" ones being $\{0\}$ and $\mathbb{C}^n$). Thus, the gist of Burnside's Theorem is in its "only if" part.

Remark 3. For an explicit example, take G to be the dihedral group G generated by the rotation $r = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ and the reflection $s = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. It can be seen that G has no invariant subspaces in $\mathbb{C}^2$, and in fact, $r, s, rs = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, together with the identity matrix clearly form a basis of $\mathbf{M}_2(\mathbb{C})$.

Remark 4. Burnside's Theorem (and its subsequent generalization by Frobenius and Schur) proved to be a fundamental result in the representation theory of groups, and has appeared in many books on that subject. From a ring-theoretic perspective, these yield a more general result, nowadays also called Burnside's Theorem, which can be formulated as follows.

Theorem 5. *Let A be a subring of $\mathbf{M}_n(\mathbb{C})$ containing all scalar matrices. If A has no nontrivial invariant subspaces in $\mathbb{C}^n$, then $A = \mathbf{M}_n(\mathbb{C})$.*

Remark 6. Note that Theorem 1 follows from Theorem 2 by applying the latter to the $\mathbb{C}$-span of the group G.

For the rest of this article, let $V = \mathbb{C}^n$, $R = \mathbf{M}_n(\mathbb{C})$, and let $A \subseteq R$ be a subring satisfying the hypotheses of Theorem 5.

Lemma 7. *Any $g \in R$ commuting with all $f \in A$ is a scalar matrix.*

Proof. Let $\lambda \in \mathbb{C}$ be an eigenvalue of g, and $E \subseteq V$ be the associated eigenspace $\{v \in V : gv = \lambda v\}$. For any $f \in A$, $fg = gf$ clearly implies that $f(E) \subseteq E$. Since $E \neq 0$, we have $E = V$, and so $g = \lambda I$. □

Lemma 8. *Let $v \in V$ and W be a subspace of V. Assume that the following holds: If any $f \in A$ is zero on W, (that is, $f(W) = 0$) then $f(v) = 0$. It then follows that $v \in W$.*

Proof. We proceed by induction on $\dim W \geq 0$. The case $\dim W = 0$ is clear, in view of the fact that $I_n \in A$. In case $\dim W > 0$, write W as a sum of a proper subspace W_0 and a line $\mathbb{C}w$ where $w \notin W_0$, and consider the $\mathbb{C}$-subspace

$$H = \{h \in A : \ h(W_0) = 0\} \subseteq A.$$

By the inductive hypothesis, $H(w) \neq 0$. Since $AH \subseteq H$, we have $A(H(w)) \subseteq H(w)$, and so $H(w) = V$. Now define a linear map $g : V \to V$ by $g(h(w)) = h(v)$ (for any $h \in H$). To check well-definition, suppose $h(w) = 0$ (for some $h \in H$). Then $h(W) = 0$, and so $h(v) = 0$ by assumption. Now g commutes with any $f \in A$, since

$$(gf)(h(w)) = g((fh)(w)) = (fh)(v) = f(g(h(w)) = (fg)(h(w))$$

for any $h \in H$. Therefore, by Lemma 7, $g = aI$ for some $a \in \mathbb{C}$. Thus, $h(v) = g(h(w)) = ah(w)$, and so $h(v - aw) = 0$ for any $h \in H$. By the inductive hypothesis again, we have $v - aw \in W_0$, and hence $v \in W$ as desired. □

Proof. It suffices to show that A contains all the matrix units E_{ij}. For ease of notation, assume that $j = 1$. Let $e_1, ..., e_n \in V$ be the standard basis. Let $H = \{h \in A : \ h(e_2) = \cdots = h(e_n) = 0\}$. By Lemma 8, $H(e_1) \neq 0$, and as before, $H(e_1)$ is invariant under A. Therefore, $H(e_1) = V$; in particular, there exists $h \in H$ such that $h(e_1) = e_i$. We have then $h = E_{i1} \in A$, as desired. □

We give an alternative proof of Theorem 5 following [1].

Theorem 9 (Burnside)**.** *Let V be a finite dimensional vector space over an algebraically closed field. Assume* $\dim V > 1$. *If $\mathcal{A}$ is an algebra of linear maps of V such that $\mathcal{A}$ leaves no nontrivial subspace invariant, then* $\mathcal{A} = \mathrm{End}(V)$.

Proof. We claim that $\mathcal{A}$ contains at least one element of rank 1. Let $A \in \mathcal{A}$ of minimal nonzero rank. If rank of A is not one, then there exist v_1 and v_2 such that Av_1 and Av_2 are linearly independent. Since $\mathcal{A}v = V$ for any nonzero v, we can find $B \in \mathcal{A}$ such that $BAv_1 = v_2$. Hence $ABAv_1$ and Av_1 are linearly independent so that $ABA - \lambda A \neq 0$ for any scalar λ. Since the field is algebraically closed, there exists λ such that $AB - \lambda I$ is not invertible on AV.But then, the rank of $(AB - \lambda I)A$ is less than that of A and is not zero—a contradiction.

Since any linear map is the sum of linear maps of rank one, it suffices to show that all rank one maps are in $\mathcal{A}$.

By the first paragraph, we know there exists at least one map of rank one, say, T. Let $Tv = f(v)v_0$ for some fixed $v_0 \in V$ and a fixed nonzero linear functional on V. Note that if

$A \in \mathcal{A}$ and if we set $\psi(v) := f(Av)$, then $\psi \in A$, for, $\psi = T \circ A$. Now, if v is annihilated by all such ψ, then, since $\mathcal{A}v = V$ and $f \neq 0$, we conclude that $v = 0$. This means that all T such that $Tx = g(x)v_0$ for some nonzero linear functional g lie in $\mathcal{A}$. Now, any rank one map B is of the form $Bx = g(x)v$ for some linear functional g and a fixed vector $v \in V$. Consider $Ax := g(x)v_0$. Let $A_0 \in \mathcal{A}$ be such that $A_0 v_0 = v$. Then $A_0 Ax = A_0(g(x)v) = g(x)v$. That is, $\mathcal{A}$ contains all rank one maps. □

As applications, we shall indicate proofs of some results related th the restricted Burnside problem.

Remark 10. Let G be finitely generated group such that there exists $N \in \mathbb{N}$ such that $x^N = e$ for all $x \in G$. The restricted Burnside problem poses the following question:

Is G finite?

We give two results which are in the affirmative.

Lemma 11. *Let $G \subset GL(n, \mathbb{C})$ be a subgroup. Assume that the only subspaces invariant under G are the zero subspace and $\mathbb{C}^n$. (That is, if $W \subset \mathbb{C}^n$ is a vector subspace such that $gw \in W$ for all $g \in G$ and $w \in W$, then $W = \{0\}\}$ or $W = \mathbb{C}^n$.) Assume further that the image $\mathrm{Tr}(G) \subset \mathbb{C}$ is a finite subset with r elements. Then G is finite with at most r^{n^2} elements.*

Proof. By Burnside's theorem, the set of finite linear combination of elements from G is $M(n, \mathbb{C})$. Hence we can find $g_k \in G$, $1 \leq k \leq n^2$ which form a basis of $M(n, \mathbb{C})$. Consider the map $\tau \colon M(n, \mathbb{C}) \to \mathbb{C}$ given by

$$\tau(\sigma) := (\mathrm{Tr}(\sigma g_1), \ldots, \mathrm{Tr}(\sigma g_{n^2})), \qquad \sigma \in M(n, \mathbb{C}).$$

The map τ is obviously linear. We show that its kernel is trivial. Let $\tau(\sigma) = 0$. It follows that $\mathrm{Tr}(\sigma a) = 0$ for any $a \in M(n, \mathbb{C})$. If we take $a = E_{ij}$, a matrix unit, then we deduce that that (ij)-th entry of σ is zero. Consequently, $\sigma = 0$. So we conclude that τ is one-one and hence a linear isomorphism. Now for any $g \in G$, each of the n^2-coordinates of $\tau(g)$ has r choices. Thus, $|\tau(G)| \leq r^{n^2}$. □

Theorem 12 (Burnside)**.** *Let $G \subset GL(n, \mathbb{C})$. Assume that there exists $N \in \mathbb{N}$ such that $g^N = I$ for any $g \in G$. Then $|G| \leq N^{n^2}$.*

Proof. If $n = 1$, the result is clear. So, we assume that $n \geq 2$. Since $x^N = 1$ for all $x \in G$, any eigenvalue of x is an N-th root of unity. In particular, $\mathrm{Tr}(x)$, being the sum of the eigenvalues of x, takes at most $r = N^n$ values in $\mathbb{C}$.

If there exists no nontrivial G-invariant subspace of $\mathbb{C}^n$, it follows from the last lemma that $|G| \leq r^{n^2} = N^{n^3}$.

If $W \subset \mathbb{C}^n$ is a nontrivial G-invariant subspace, then we choose a basis of W and extend it to a basis of $\mathbb{C}^n$. With respect to this basis, G will consist elements of the form $\begin{pmatrix} g_1 & h \\ 0 & g_2 \end{pmatrix}$ where g_1 is matrix of size $n_1 = \dim W$ and g_2 of size $n_2 = n - \dim W$. If we set G_i to be the set of such g_i that arise in the above representation of $g \in G$, it is easy to check that G_i

are subgroups with the property that $x^N = 1$ for $x \in G_i$, $i = 1, 2$. We can apply indction on n to conclude that $|G_i| \leq N^{n_i^3}$. The map $g \mapsto (g_1, g_2)$ of G into $G_1 \times G_2$ is verified to be a homomorphism. We claim that this map is one-one. For, if $g = \begin{pmatrix} g_1 & h \\ 0 & g_2 \end{pmatrix}$ lies in the kernel, then $g_1 = I$ and $g_2 = I$ and

$$I = g^N = \begin{pmatrix} I & h^N \\ 0 & I \end{pmatrix} = \begin{pmatrix} I & Nh \\ 0 & I \end{pmatrix}.$$

We conclude that $h = 0$. Thus the above map is one-one and so

$$|G| \leq |G_1| \cdot |G_2| \leq N^{n_1^3} \cdot N^{n_2^3} \leq N^{n^3}.$$

□

Theorem 13. *A subgroup $G \subset GL(n, \mathbb{C})$ is finite iff it has a finite number of conjugacy classes*

Proof. Since the trace is a constant on conjugacy classes, we deduce that $\mathrm{Tr}(G)$ is finite. If there exist no nontrivial G-invariant subspaces, then the result follows from Lemma 11.

Let there exist a nontrivial G-invariant subspace. We use the notation of the proof of the last theorem. We leave it as an exercise to show that G_i have only finitely many conjugacy classes. By induction, we conclude that G_i are finite. The kernel H of the map $g \mapsto (g_1, g_2)$ is normal in G. It consists of elements of the form $\begin{pmatrix} I & h \\ 0 & I \end{pmatrix}$. This normal subgroup is abelian:

$$\begin{pmatrix} I & h \\ 0 & I \end{pmatrix} \begin{pmatrix} I & h' \\ 0 & I \end{pmatrix} = \begin{pmatrix} I & h + h' \\ 0 & I \end{pmatrix} = \begin{pmatrix} I & h' \\ 0 & I \end{pmatrix} \begin{pmatrix} I & h \\ 0 & I \end{pmatrix}.$$

We claim that any $g \in H$ has only finitely many G-conjugates. For, note that $[G : H] \leq |G_1| \cdot |G_2| < \infty$. Hence the index of the centralizer $C_G(g)$ **in** G is also finite. Since there are only finitely many G-conjugacy classes, H must be finite. (The reader should justify this.) Since we have already noted that $[G : H] < \infty$, it follows that G is finite. □

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References:

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(3) T.Y. Lam, *A first course in noncommutative rings*, Springer-Verlag, 1991, GTM 131