Examples of Complete ON Bases in Hilbert Spaces

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1. **Fourier Series and a Complete ON Basis for $L^2[-\pi,\pi]$.** A good reference, which we follow closely, is Rudin's *Real and Complex Analysis*, especially §4.23–§4.26.

   (a) A *trigonometric polynomial* $p(x)$ is an expression of the form $p(x) = \sum_{|k|\le n} c_k e^{ikx}$. It is said to be of *degree* n if at least one of $|c_{-n}|$ and $|c_n|$ is non-zero where $c_k \in \mathbb{C}$. Note that p is a continuous of *period* 2π (in the sense that $p(x+2\pi) = p(x)$ for all $x \in \mathbb{R}$). c_k is given by
$$c_k = \frac{1}{2\pi}\int_{-\pi}^{\pi} p(x)e^{-ikx}dx,$$
   since
$$\frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-irx}dx = \begin{cases} 0 & \text{if } r \neq 0 \\ 1 & \text{if } r = 0. \end{cases}$$
   A trigonometric series is of the form $\sum_{-\infty}^{\infty} c_k e^{ikx}$ (just a formal expression; no assumption is made on the convergence of the series).
   If $f \in L^1[-\pi,\pi]$, then the *Fourier series* of f is the trigonometric series
$$\sum_{-\infty}^{\infty} c_k e^{ikx} \text{ where } c_k := \widehat{f}(k) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)e^{-ikx}dx.$$
   We then write $f \sim \sum \widehat{f}(k)e^{ikx}$. $\widehat{f}(k)$ are called the *Fourier coefficients* of f. Note that $|\widehat{f}(k)| \le \|f\|_{L^1[-\pi,\pi]} = \|f\|_{L^1}$.
   Let $s_n(f,x) := \sum_{|k|\le n} \widehat{f}(k)e^{ikx}$ be the n-th symmetric partial sum of the Fourier series of $f \in L^1[-\pi,\pi]$.

   (b) Prove the **Riemann Lebesgue Lemma:** For $f \in L^1[-\pi,\pi]$, $\lim_n \widehat{f}(n) = 0$. *Hint:* Prove this for a characteristic function of an interval $[a,b] \subseteq [-\pi,\pi]$. Use the fact that step functions are dense in $L^1[-\pi,\pi]$.

   (c) A sequence $\{K_n\}$ of real valued continuous functions in $[-\pi,\pi]$ (with period 2π) is called an *approximate identity* on $[-\pi,\pi]$ if it has the following three properties:
      (i) K_n is periodic and $K_n \ge 0$.
      (ii) $\frac{1}{2\pi}\int_{-\pi}^{\pi} K_n = 1$.

(iii) Given $\varepsilon > 0$ and $\delta > 0$, there exists N such that if $n \geq N$ then

$$\left(\int_{-\pi}^{-\delta} + \int_{\delta}^{\pi}\right) K_n < \varepsilon$$

Geometrically, (iii) says that the area under the graph of K_n accumulates around the point 0 as $n \to \infty$.

(d) Let $\{K_n\}$ be an approximate identity on $[-\pi, \pi]$. Let f be a continuous function as $[-\pi, \pi]$ of period 2π. Then $f_n(x) := f * K_n(x) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)K_n(x-t)dt$ converges uniformly to f on $[-\pi, \pi]$.

$$\begin{aligned} |f_n(x) - f(x)| &= \frac{1}{2\pi}\left|\int f(t)K_n(x-t)dt - \int f(x)K(t)dt\right| \\ &\leq \frac{1}{2\pi}\left|\int [f(x+t) - f(x)]K(t)dt\right| \\ &\leq \frac{1}{2\pi}\int |[f(x+t) - f(x)]|K(t)dt \\ &\leq \frac{1}{2\pi}\left(\int_{-\pi}^{-\delta} + \int_{\delta}^{\pi} + \int_{-\delta}^{\delta}\right), \end{aligned}$$

where δ is chosen by uniform continuity of f. The first two terms are estimated using the bound for f and property (iii) of an approximate identity. The third is estimated using the uniform continuity of f.

(e) We now give an explicit approximate identity. Let $K_n(t) := C_n\left(\frac{1+\cos t}{2}\right)^n$ where C_n is chosen so that $\frac{C_n}{2\pi}\int_{-\pi}^{\pi} K_nt)\, dt = 1$.

(f) Observe that K_n is even, decreasing on $[0, \pi]$ and that it satisfies the first two properties of an approximate identity (as in Item 7j).

(g) We need an upper bound for C_n, that is, a lower bound for $\int_{-\pi}^{\pi} K_n(t)\, dt$. Enough to consider the integral over $[0, \pi]$.

$$\begin{aligned} \int_0^{\pi} K_n(t)\, dt &\geq \int_0^{\pi} K_n(t) \sin t\, dt && (1) \\ &= \int_0^1 u^n du = \frac{1}{n+1}. && (2) \end{aligned}$$

(h) To verify that the third condition in Item 7j also holds, we establish a stronger property: Given $0 < \delta < \pi$,

$$M_n(\delta) := \sup_{t \geq \delta} K_n(t) \to 0.$$

Since K_n is decreasing we have, for $t \geq \delta$,

$$K_n(t) \leq K_n(\delta) \equiv C_n\left(\frac{1+\cos\delta}{2}\right)^n \geq (n+1)r^n,$$

where $r := \frac{1+\cos\delta}{2}$. As $(n+1)r^n \to 0$ as $n \to \infty$ (why?), the result follows.

(i) Note that $p_n(t) := \frac{1}{2\pi}\int_{-\pi}^{\pi} f(s)K(s-t)\,dt$ is a trigonometric polynomial. So we have shown that the vector subspace/algebra of trigonometric polynomials is dense in $C(\mathbb{T}), \|\ \|_\infty)$.

(j) Let $f \in \mathcal{C}([-\pi,\pi])$ be periodic. Assume $\widehat{f}(k) = 0$ for all $k \in \mathbb{Z}$. Then $f = 0$. *Hint:* Assume f to be real. Then $\int fp = 0$ for all trigonometric polynomials p. Use the last exercise to conclude that $\int f^2 = 0$.

(k) Show that the set of continuous functions in $\mathcal{C}[-\pi,\pi]$ which are periodic, i.e., $f(+\pi) = f(-\pi)$ is dense in $L^2[-\pi,\pi]$. *Hint:* Recall that $\mathcal{C}[-\pi,\pi]$ is dense in $L^2[-\pi,\pi]$. Given $g \in \mathcal{C}[-\pi,\pi] \subseteq L^2[-\pi,\pi]$, consider

$$g_n = \begin{cases} g(t) & \text{if } -\pi \le t \le t_n := \pi - \frac{1}{n^2} \\ g(t_n) - [g(-\pi) - g(t_n)]\left(\frac{t-t_n}{\pi - t_n}\right) & \text{if } t \in [t_n, \pi]. \end{cases}$$

(l) Show that the set of periodic continuous functions $C(\mathbb{T})$ is not dense in $(C[-\pi,\pi], \|\ \|_\infty)$.

(m) Is the set of trigonometric polynomials dense in $L^2[-\pi,\pi]$? If so, the last item is ‘obvious’.

(n) Let $f \in L^2[-\pi,\pi]$ be such that $\widehat{f}(n) = 0$ for all $n \in \mathbb{N}$. Then $f = 0$ a.e. *Hint:* Assume f to be real. The hypothesis implies that $\int fg = 0$ for any periodic continuous function g. Use Item 7o. Or use Item 7q.

(o) For a lot of examples of approximate identities and their uses in analysis, refer to my artilce on “Approximate Identities”.

2. The set $\{e_n(z) := \sqrt{n+1} z^n\}$ is an O.N. set in $L^2_H(\mathbb{D})$. We show that it is complete by showing that if $f \in L^2_H(\mathbb{D}, dA)$ is such that $\int_{\mathbb{D}} f(z)\overline{z}^n dA = 0$, then $f = 0$.

   If f is holomorphic in $\mathbb{D}$, then we have a power series expansion of f in $\mathbb{D}$ (Why?):

$$f(z) = \sum_{m \in \mathbb{Z}_+} c_n z^m, z \in \mathbb{D}.$$

   The power series is uniformly convergent on compact subsets of $\mathbb{D}$, in particular, uniformly convergent on $B(0,r) = r\mathbb{D}$. Hence

$$\int_{r\mathbb{D}} f(z)\overline{z}^n \, dA = \sum_m c_m \int_{r\mathbb{D}} z^m \overline{z}^n = c_m \pi \frac{r^{2(n+1)}}{n+1}.$$

   Note that $f(z)\overline{z}^n \in L^2(\mathbb{D}) \subset L^1(\mathbb{D})$. We use DCT to take the limit $r \to 1$ and arrive at $\int_{\mathbb{D}} f(z)\overline{z}^n \, dA = \pi \frac{c_n}{n+1}$. The assumption $f \perp z^n$ now leads to us to conclude that $c_n = 0$. Hence $f = 0$.

3. **Laguerre Functions and a Complete ON Basis for $L^2[0,\infty)$.**

   (a) The Laguerre polynomials are defined by the formulas

$$L_n(x) = e^x D^n(x^n e^{-x})(-1)^n x^n + \sum_{k=0}^{n-1} (-1)^k \binom{n}{k} n(n-1)\cdots(k+1)x^k,$$

   for $n \in \mathbb{Z}_+$. Note that L_n is a polynomial of degree n.

   (b) Laguerre functions ϕ_n are defined by

$$\phi_n(x) = \frac{1}{n!} e^{-x/2} L_n(x), n \in \mathbb{Z}_+.$$

   (c) We claim that $\{\phi_n : n \in \mathbb{Z}_+\}$ is an orthogonal subset of $L^2(0,\infty)$.
   Let $m < n$. If we show that $\int_0^\infty x^k L_n(x) e^{-x} \, dx = 0$ for all $k < n$, it will follow that $\langle L_m, L_n \rangle = 0$. For, L_m is a polynomial of degree m and $m < n$.
   To prove this, we integrate $\int_0^\infty x^k L_n(x) e^{-x} \, dx$ by parts k-times:

$$\int_0^\infty x^k L_n(x) e^{-x} \, dx = \int_0^\infty x^k D^n(x^n e^{-x}) \, dx = (-1)^k k! \int_0^\infty D^{n-k}(x^n e^{-x}) \, dx = 0.$$

   (Why? The integrand has a primitive, as $n - k > 0$ and the primitive takes the value 0 at the end points.)

   (d) $\|\phi_n\|_2 \equiv \|\phi_n\|_{L^2(0,\infty)} = 1$.
   We use the last item and integrate by parts n-times.

$$\begin{aligned}
\left\| \frac{1}{n!} e^{-x/2} L_n \right\|^2 &= \frac{1}{(n!)^2} \int_0^\infty L_n(x) L_n(x) e^{-x} \, dx \\
&= \frac{1}{(n!)^2} \int_0^\infty (-1)^n x^n D^n(x^n e^{-x}) \, dx \\
&= \frac{1}{(n!)^2} (n!)^2 \int_0^\infty x^n e^{-x} \, dx \\
&= 1.
\end{aligned}$$

(e) The difficult step is to show that the ON set is a complete ON basis. If $f \in L^2(0,\infty)$ is such that $\langle f, \phi_n \rangle = 0$ for $n \in \mathbb{Z}_+$, then $f = 0$ a.e.

Let $g(x) = f(x)e^{-x/2}$. Note that $\int_0^\infty g(x) x L_n(x)\, dx = 0$. (Why?) This implies that $\int_0^\infty x^n g(x) = 0$ for $n \in \mathbb{Z}_+$. (Why?) Enough to show that $g = 0$ a.e. We do this in a series of steps.

i. Consider $F(s) := \int_0^\infty e^{-sx} g(x)\, dx$. We show that F is holomorphic on $\operatorname{Re} s > 0$, zero for $0 < s < 1/2$ and hence $F = 0$. Since F is the Laplace transform of g, and since Laplace transform is one-one, it will follow that $g = 0$ a.e.

ii. We show that F is holomorphic in $\operatorname{Re} s > 0$. To do this, we use Lagrange's formula for differentiation under the integral sign. Fix t with $\operatorname{Re} t > 0$. Then there exists $r > 0$ and $\delta > 0$ such that for all $z \in B(t,r)$ we have $\operatorname{Re} z > \delta$. Then there exists $M > 0$ such that

$$|\frac{\partial e^{-sx}}{\partial s} g(x)| \le |-xe^{-sx} g(x)| \le M|g(x)|.$$

(Why? $x^n e^{-x} \to 0$ as $x \to \infty$.) Since $|g| \in L^1(0,\infty)$ (why?), we can differentiate under the integral sign and get $F'(s) = -\int_0^\infty x e^{-sx} g(x)\, dx$. Hence F is holomorphic.

iii. Using Holder's inequality and the definition of gamma function, we obtain

$$\int_0^\infty |x^n g(x)|\, dx = \int_0^\infty |x^n e^{-x/2} f(x)|\, dx \le \|f\|\, (2n!)^{1/2}.$$

iv. In particular, for $s \ge 0$, we have

$$\int_0^\infty \frac{s^n}{n!} |g(x)|\, dx \le \frac{s^n}{n!} \|f\|\, (2n!)^{1/2} =: a_n, \text{ say.}$$

v. The ratio test shows that the series $\sum_{n=0}^\infty a_n$ is convergent for $0 < s < 1/2$. Hence by DCT it follows that

$$\sum_{n=0}^\infty (-1)^n \frac{(sx)^n}{n!} g(x) = e^{-sx} g(x) \text{ in } L^1, 0 < s < 1/2.$$

vi. It follows

$$F(s) = \sum_{n=0}^\infty (-1)^n \frac{s^n}{n!}, 0 < s < 1/2.$$

vii. Hence $F(s) = 0$ for $0 < s < 1/2$. By the identity theorem for holomorphic functions, $F = 0$ on $\operatorname{Re} s > 0$. Hence $g = 0$ a.e.

4. **(An Extension of DCT.)** Let $(X, \mathcal{B}, \mu)$ be a measure space, $J \subset \mathbb{R}$ an interval.
(i) Assume that $f\colon X \times J \to \mathbb{R}$ be such that $f_t(x) := f(x,t)$ is measurable for each $t \in J$.
(ii) Assume further that there exists $g \in L^1$ such that $|f_t(x)| \le g(x)$ a.e.
(iii) For t_0, a cluster point of J, assume that $\lim_{t \to t_0} f_t(x) = h(x)$ exists a.e.
Then h is integrable and

$$\lim_{t \to t_0} \int f(x,t)\, d\mu(x) = \int \lim_{t \to t_0} f(x,t)\, d\mu(x) = \int h(x)\, d\mu(x).$$

5. **(Differentiation under the integral sign.)** Let $(X, \mathcal{B}, \mu)$ be a measure space. Let $f : X \times (a, b) \to \mathbb{R}$ be a function such that $f_t \in L^1$ for $t \in (a, b)$. Assume that for $t_0 \in (a, b)$, the partial derivative $\frac{\partial f}{\partial t}(x, t_0)$, defined as

$$\frac{\partial f}{\partial t}(x, t_0) = \lim_{t \to t_0} \frac{f(x, t) - f(x, t_0)}{t - t_0},$$

exists a.e. Assume further that there exists $g \in L^1$ and $\delta > 0$ such that

$$|\frac{f(x, t) - f(x, t_0)}{t - t_0}| \leq g(x) a.e.$$

Then
(a) $\frac{\partial f}{\partial t}(x, t_0)$ is in L^1
(b)the function $F : (a, b) \to \mathbb{R}$ defined as $F(t) := \int f(x, t) d\mu(x)$ is differentiable at t_0 and we have

$$F'(t_0) = \int \frac{\partial f}{\partial t}(x, t_0) d\mu(x).$$

6. **Laplace Transform.** Let $\mathcal{L} : L^1[0, \infty) \to C_b[0, \infty)$ be defined by

$$\mathcal{L}f(s) := \int_0^\infty e^{-st} f(t)\, dt, \text{ for } s \geq 0.$$

$\mathcal{L}f$ is called the Laplace transform of f. We list some of the properties of the Laplace transform.

(a) $e^{-st} f(t)$ is measurable. Since $|e^{-st} f(t)| \leq |f|$ for $s \geq 0$, it is integrable.

(b) $|\mathcal{L}f(s)| \leq \|f\|_1$ for $s \geq 0$ and hence $\|\mathcal{L}f\|_\infty \leq \|f\|_1$. Hence $\mathcal{L}f$ is bounded.

(c) Note that $\lim_{s \to s_0} e^{st} f(t) = e^{-s_0 t} f(t)$. Using the extension of DCT (Item 4), we see that

$$\begin{aligned} \lim_{s \to s_0} \mathcal{L}f(s) &= \lim_{s \to s_0} \int_0^\infty e^{-st} f(t)\, dt \\ &= \int_0^\infty \lim_{s \to s_0} e^{-st} f(t)\, dt \\ &= \mathcal{L}f(s_0). \end{aligned}$$

We therefore conclude that $\mathcal{L}$ maps $L^1[0, \infty)$ into $C_b[0, \infty)$.

(d) In fact, the last argument shows that $\mathcal{L}f$ is a continuous function vanishing at infinity.

(e) Thus we have shown $\mathcal{L} : (L^1[0, \infty), \|\ \|_1) \to (C_b[0, \infty), \|\ \|_\infty)$ is a continuous linear map.

(f) We now show that $\mathcal{L}$ is one-one. Let $f \in L^1[0, \infty)$ be such that $\mathcal{L}f(s) = 0$ for $s \geq 0$. Let $u = e^{-t}$ so that the integral for $\mathcal{L}f(s)$ becomes $\int_0^1 u^{s-1} f(-\log u)\, du = 0$ for any $s \geq 0$, in particular for any $s \in \mathbb{Z}_+$. Using the density of the space of polynomials in $L^1[0, 1]$, we deduce that $f(-\log u) = 0$ a.e. on [0,1]. Since any Lipschitz function maps sets of measure zero to sets of measure zero, it follows that $f = 0$ a.e. (Work out the details!)

7. The aim of this item is to prove the following result.

Theorem 1. *$\{e^{inx} : n \in \mathbb{Z}\}$ is a complete O.N. basis for $L^2[-\pi, \pi]$.*

(a) A *trigonometric polynomial* $p(x)$ is an expression of the form $p(x) = \sum_{|k|\leq n} c_k e^{ikx}$. It is said to be of *degree* n if at least one of $|c_{-n}|$ and $|c_n|$ is non-zero where $c_k \in \mathbb{C}$. Note that p is a continuous of *period* 2π (in the sense that $p(x + 2\pi) = p(x)$ for all $x \in \mathbb{R}$). c_k is given by

$$c_k = \frac{1}{2\pi}\int_{-\pi}^{\pi} p(x)e^{-ikx}dx,$$

since

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-irx}dx = \begin{cases} 0 & \text{if } r \neq 0 \\ 1 & \text{if } r = 0. \end{cases}$$

A trigonometric series is of the form $\sum_{-\infty}^{\infty} c_k e^{ikx}$ (just a formal expression; no assumption is made on the convergence of the series).

If $f \in L^1[-\pi, \pi]$, then the *Fourier series* of f is the trigonometric series

$$\sum_{-\infty}^{\infty} c_k e^{ikx} \text{ where } c_k := \widehat{f}(k) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)e^{-ikx}dx.$$

We then write $f \sim \sum \widehat{f}(k)e^{ikx}$. $\widehat{f}(k)$ are called the *Fourier coefficients* of f. Note that $|\widehat{f}(k)| \leq \|f\|_{L^1[-\pi,\pi]} = \|f\|_{L^1}$.

Let $s_n(f, x) := \sum_{|k|\leq n} \widehat{f}(k)e^{ikx}$ be the n-th symmetric partial sum of the Fourier series of $f \in L^1[-\pi, \pi]$.

(b) Prove the **Riemann Lebesgue Lemma:** For $f \in L^1[-\pi, \pi]$, $\lim_n \widehat{f}(n) = 0$. *Hint:* Prove this for a characteristic function of an interval $[a, b] \subseteq [-\pi, \pi]$. Use the fact that step functions are dense in $L^1[-\pi, \pi]$.

(c) Derive the following expression for $s_n(f, x)$:

$$s_n(f, x) = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\left(\frac{1}{2}\sum_{|k|\leq n} e^{ik(x-t)}\right) dt.$$

Let $D_n(x) := \frac{1}{2}\sum_{-n}^{n} e^{ikx}$. Then $D_n(x)$ is called the *n-th Dirichlet kernel* and $s_n(f, x) = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)D_n(x - t)dt$.

(d) Sum the geometric series $D_n(x)$ to get $D_n(x) = \dfrac{1}{2}\dfrac{\sin(n + 1/2)x}{\sin(x/2)}$.

(e) Given a sequence (α_n) of complex numbers, we say that α_n converges to α in *Ceasaro means* (and write this as $C{-}\lim \alpha_n = \alpha$) if the averages $a_n := \dfrac{\alpha_1 + \cdots + \alpha_n}{n} \to \alpha$.

If $\lim \alpha_n = \alpha$, then $C - \lim \alpha_n = \alpha$. The converse however is not true.

(f) Define the Ceasaro summability of a series as: If $\sum \alpha_n$ is given, define $s_n := \sum_1^k \alpha_k$ and $\sigma_n := \frac{1}{n}\sum_1^k s_k$. We say $\sum \alpha_n$ is *Ceasaro summable* to σ if $\lim \sigma_n = \sigma$. We then write $C - \sum \alpha_n = \sigma$.

If $\sum \alpha_n = s$, then $C - \sum \alpha_n = s$. However, converse is not true. *Hint:* Take $\alpha_n = z^n$, for $|z| = 1$, with $z \neq 1$.

(g) Given f, we let

$$\sigma_n(f,x) := \frac{s_0(f,x)+\cdots+s_n(f,x)}{n+1}, \quad n \in \mathbb{Z}^+$$

called the *n-th Ceasaro sum* of f.

(h) Show that for $f \in L^1[-\pi,\pi]$,

$$\sigma_n(f,x) := \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)K_n(x-t)dt$$

where $K_n(x) := \frac{1}{n+1}\sum_{k=0}^{n} D_k(x)$ is the *n-th Fejer kernel.*

(i) Show that

$$K_n(x) = \frac{1}{2(n+1)} \cdot \frac{\sin^2(\frac{(n+1)x}{2})}{\sin^2\frac{x}{2}}.$$

Hint:

$$\begin{aligned}\sum_{k=0}^{n} D_k(x) &= \sum_{k=0}^{n}\frac{\sin(k+\frac{1}{2})x}{2\sin\frac{x}{2}}\\ &= \frac{1}{2\sin x/2}\times \operatorname{Im}\left(\sum_{k=0}^{n} e^{i(k+1/2)x}\right)\\ &= \frac{1}{2\sin x/2}\times \operatorname{Im}\left(e^{ix/2}\frac{1-e^{i(n+1)x}}{1-e^{ix}}\right).\end{aligned}$$

(j) The sequence $\{K_n\}$ of Fejer kernels has the following properties:
   (i) K_n is periodic and $K_n \geq 0$.
   (ii) $\frac{1}{2\pi}\int_{-\pi}^{\pi} K_n = 1$.
   (iii) Given $\varepsilon > 0$ and $\delta > 0$, there exists N such that if $n \geq N$ then

$$\left(\int_{-\pi}^{-\delta} + \int_{\delta}^{\pi}\right) K_n < \varepsilon$$

Hint: To prove (iii) observe that

$$\frac{1}{n}\int_{\delta}^{\pi}\frac{\sin^2 nt/2}{\sin^2 t/2}dt \leq \frac{1}{n}\int_{\delta}^{\pi}\frac{1}{\sin^2 t/2}dt$$

and the last integral is a real number.

Geometrically, (iii) says that the area under the graph of K_n accumulates around the point 0 as $n \to \infty$.

(k) A sequence $\{K_n\}$ of real valued continuous functions in $[-\pi,\pi]$ (with period 2π) is called an *approximate identity* on $[-\pi,\pi]$ if it has the three properties listed in the last exercise. Thus the sequence $\{K_n\}$ of Fejer kernels is an approximate identity.

(l) Let $\{K_n\}$ be an approximate identity on $[-\pi, \pi]$. Let f be a continuous function as $[-\pi, \pi]$ of period 2π. Then $f_n(x) := f * K_n(x) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)K_n(x-t)dt$ converges uniformly to f on $[-\pi, \pi]$.

$$\begin{aligned} |f_n(x) - f(x)| &= \frac{1}{2\pi}|\int f(t)K_n(x-t)dt - \int f(x)K(t)dt| \\ &\leq \frac{1}{2\pi}|\int [f(x+t) - f(x)]K(t)dt| \\ &\leq \frac{1}{2\pi}\int |[f(x+t) - f(x)]|K(t)dt \\ &\leq \frac{1}{2\pi}\left(\int_{-\pi}^{-\delta} + \int_{\delta}^{\pi} + \int_{-\delta}^{\delta}\right), \end{aligned}$$

where δ is chosen by uniform continuity of f. The first two terms are estimated using the bound for f and property (iii) of an approximate identity. The third is estimated using the uniform continuity of f.

(m) Let $f \in \mathcal{C}([-\pi, \pi])$ be periodic. Then given $\varepsilon > 0$, there exists a trigonometric polynomial p such that $|f(x) - p(x)| < \varepsilon$ for $x \in [-\pi, \pi]$. *Hint:* $p = \sigma_n(f, x)$ for sufficiently large n.

(n) Let $f \in \mathcal{C}([-\pi, \pi])$ be periodic. Assume $\widehat{f}(k) = 0$ for all $k \in \mathbb{Z}$. Then $f = 0$. *Hint:* Assume f to be real. Then $\int fp = 0$ for all trigonometric polynomials p. Use the last exercise to conclude that $\int f^2 = 0$.

(o) Show that the set of continuous functions in $\mathcal{C}[-\pi, \pi]$ which are periodic, i.e., $f(+\pi) = f(-\pi)$ is dense in $L^2[-\pi, \pi]$. *Hint:* Recall that $\mathcal{C}[-\pi, \pi]$ is dense in $L^2[-\pi, \pi]$. Given $g \in \mathcal{C}[-\pi, \pi] \subseteq L^2[-\pi, \pi]$, consider

$$g_n = \begin{cases} g(t) & \text{if } -\pi \leq t \leq t_n := \pi - \frac{1}{n^2} \\ g(t_n) - [g(-\pi) - g(t_n)]\left(\frac{t-t_n}{\pi - t_n}\right) & \text{if } t \in [t_n, \pi]. \end{cases}$$

(p) Show that the set of periodic continuous functions $C(\mathbb{T})$ is not dense in $(C[-\pi, \pi], \| \; \|_\infty)$.

(q) Show that the set of trigonometric polynomials is dense in $L^2[-\pi, \pi]$.

(r) Let $f \in L^2[-\pi, \pi]$ be such that $\widehat{f}(n) = 0$ for all $n \in \mathbb{N}$. Then $f = 0$ a.e. *Hint:* Assume f to be real. The hypothesis implies that $\int fg = 0$ for any periodic continuous function g. Use Item 7o. Or use Item 7q.

8. We now indicate another proof which directly exhibits an approximate identity and proceeds as in Item 7l that $C(\mathbb{T})$ is dense in $L^2[-\pi, \pi]$.

(a) Let $K_n(t) := C_n\left(\frac{1+\cos t}{2}\right)^n$ where C_n is chosen so that $\frac{C_n}{2\pi}\int_{-\pi}^{\pi} K_n t)\, dt = 1$.

(b) Observe that K_n is even, decreasing on $[0, \pi]$ and that it satisfies the first two properties of an approximate identity (as in Item 7j).

(c) We need an upper bound for C_n, that is, a lower bound for $\int_{-\pi}^{\pi} K_n(t)\, dt$. Enough to consider the integral over $[0, \pi]$.

$$\int_0^{\pi} K_n(t)\, dt \geq \int_0^{\pi} K_n(t)\sin t\, dt \tag{3}$$

$$= \int_0^1 u^n du = \frac{1}{n+1}. \tag{4}$$

(d) To verify that the third condition in Item 7j also holds, we establish a stronger property: Given $0 < \delta < \pi$,

$$M_n(\delta) := \sup_{t \geq \delta} K_n(t) \to 0.$$

Since K_n is decreasing we have, for $t \geq \delta$,

$$K_n(t) \leq K_n(\delta) \equiv C_n \left(\frac{1 + \cos \delta}{2} \right)^n \geq (n+1) r^n,$$

where $r := \frac{1+\cos\delta}{2}$. As $(n+1)r^n \to 0$ as $n \to \infty$ (why?), the result follows.