Existence and Uniqueness of Solutions of ODE Smooth Flow Associated to a Smooth Vector-field

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1 Existence of Solutions of ODE

Proposition 1. *Let $U \subset \mathbb{R}^N$ be an open set. Let $X \colon U \to \mathbb{R}^N$ be a Lipschitz map with Lipschitz constant L: $\| X(x) - X(y) \| \le L \| x - y \|$ for all $x, y \in U$. Let $x_0 \in U$ be fixed. Let $B[x_0, r] \subset U$ and $M > 0$ be such that $\| X(x) \| \le M$ for $x \in B[x_0, r]$. Let $\varepsilon < \min\{1/L, r/M\}$. Then there exists a unique C^1 curve $x \colon [-\varepsilon, \varepsilon] \to B[x_0, r]$ which is a solution of the following initial value problem (IVP):*

$$x'(t) = X(x(t)) \qquad \text{and} \qquad x(0) = x_0. \tag{1}$$

Proof. In view of the fundamental theorem of calculus, IVP Eq. 1 is equivalent to the integral equation:

$$x(t) = x_0 + \int_0^t X(x(s))\, ds. \tag{2}$$

We solve this by Picard's method of iteration. Let $x_0(t) = x_0$ for $t \in [-\varepsilon, \varepsilon]$. Define x_n recursively:

$$x_{n+1}(t) = x_0 + \int_0^t X(x_n(s))\, ds.$$

We prove by induction that $x_n(t) \in B[x_0, r]$ for $t \in [-\varepsilon, \varepsilon]$. We have

$$\| x_1(t) - x_0 \| \le \int_0^t \| X(x_0) \| \, ds \le M \pmod t \le M\varepsilon < r.$$

Assume that we have proved the result for all $k \le n$. Now, since $x_n(s) \in B[x_0, r]$ for $s \in [-\varepsilon, \varepsilon]$, we have

$$\| x_{n+1}(t) - x_0 \| \le \int_0^t \| X(x_n(s)) \| \, ds \le M \pmod t \le M\varepsilon < r.$$

We next claim that the sequence (x_n) converge uniformly. To show this, we observe that

$$\begin{aligned}
\| x_{n+1}(t) - x_n(t) \| &\leq \int_0^t \| X(x_n(s)) - X(x_{n-1}(s)) \| \, ds \\
&\leq L \, (\operatorname{mod} t) \sup_s \| x_n(s) - x_{n-1}(s) \| \\
&\leq L^2 \, (\operatorname{mod} t)^2 \sup_s \| x_{n-1}(s) - x_{n-2}(s) \| \\
&\vdots \\
&\leq L^n \, (\operatorname{mod} t)^n \sup_s \| x_1(s) - x_0(s) \| \\
&\leq M L^n \, (\operatorname{mod} t)^{n+1}.
\end{aligned}$$

Since $M \sum_n L^n \, (\operatorname{mod} t)^{n+1} \leq M\varepsilon \sum_n (L\varepsilon)^n$ is a convergent geometric series, it follows from Weierstrass M-test, the series $\sum_n [x_n(t) - x_{n-1}(t)]$ and hence (x_n) is uniformly convergent on $[-\varepsilon, \varepsilon]$ to a continuous function $x \colon [-\varepsilon, \varepsilon] \to B[x_0, r]$. Hence appealing to the result on interchange of uniform limit and the Riemann integral for a sequence of continuous functions, we deduce that x satisfies the integral equation Eq. 2.

To prove uniqueness, let y be another solution of the IVP Eq. 1 and hence the integral equation 2 on $[-\varepsilon, \varepsilon]$. The continuous function $t \mapsto \| x(t) - y(t) \|$ assumes its maximum value, say, at t_0. We then have

$$\begin{aligned}
\| x(t_0) - y(t_0) \| &= \left\| \int_0^t [X(x(s)) - X(y(s))] \, ds \right\| \\
&\leq \int_0^t \| [X(x(s)) - X(y(s))] \| \, ds \\
&\leq L\varepsilon \sup_s \| x(s) - y(s) \| \\
&= L\varepsilon \, \| x(t_0) - y(t_0) \| .
\end{aligned}$$

Since $L\varepsilon < 1$, this inequality is true iff $\| x(t_0) - y(t_0) \| = 0$, i.e. iff $x(t) = y(t)$ for $t \in [-\varepsilon, \varepsilon]$. □

Ex. 2. Let $A \colon [-a, a] \times U \to M(n, \mathbb{R})$ be a continuous matrix valued function. Then the IVP

$$\frac{d}{dt}\psi(t, x) \equiv \psi'(t, x) = A(t, x)\psi \qquad \text{and} \qquad \psi(0, x) = I, \text{Identity} \tag{3}$$

has a unique solution in $[-\varepsilon, \varepsilon]$ for some $\varepsilon > 0$. *Hint:* Adapt the above proof. Use the operator norm $\| A \| := \max\{\| Au \| : u \in \mathbb{R}^n \text{ and } \| u \| = 1\}$. Observe that $\| AB \| \leq \| A \| \, \| B \|$.

Ex. 3. Generalize the above proposition as follows. Let $\Lambda \subset \mathbb{R}^N$ be open. Assume that $X : U \times \Lambda \to \mathbb{R}^n$ is continuous. Assume further that X is uniformly Lipschitz in x: there exists a constant L such that

$$\| X(x, \lambda) - X(y, \lambda) \| \leq L \, \| x - y \| , \quad \text{for all } x, y \in U, \lambda \in \Lambda.$$

Then there exists a unique continuous solution $x(t, \lambda)$ on $[-\varepsilon, \varepsilon] \times \Lambda$ for some suitable ε.

Keep the notation of the proposition. Let the unique solution of IVP Eq. 1 be denoted by $\gamma_{x_0}(t)$. Then γ_{x_0} is a C^1 curve from $[-\varepsilon,\varepsilon] \to B(x_0,r)$ such that $\gamma_{x_0}(0) = x_0$ and that $\gamma'_{x_0}(t) = X(\gamma_{x_0}(t))$. The next theorem shows that if we can cut down the neighbourhood of x_0 to $B(x_0,r/2)$, then we can find an $\varepsilon > 0$ such that for each $x \in B(x_0,r/2)$, we have a C^1 curve $\gamma_x \colon [-\varepsilon,\varepsilon] \to B(x_0,r)$ such that $\gamma_x(0) = x$ and $\gamma'_x(t) = X(\gamma_x(t))$ for $t \in [-\varepsilon,\varepsilon]$. More over, if we set $F(t,x) := \gamma_x(t)$ for $x \in B(x_0,r/2)$ and $(\bmod t) < \varepsilon$, then F is jointly continuous on $[-\varepsilon,\varepsilon] \times B(x_0,r/2)$. Before proving this, we need a celebrated inequality.

Lemma 4 (Gronwall Inequality)**.** *Let $f,g \colon [a,b] \to \mathbb{R}$ be nonnegative continuous functions. Assume that there is a $C \geq 0$ such that*

$$f(t) \leq C + \int_a^t f(s)g(s)\,ds.$$

Then

$$f(t) \leq C \exp\left(\int_a^t g(s)\,ds\right), \qquad \textit{for } t \in [a,b].$$

Proof. Assume that $C > 0$. Let $h(t) := C + \int_a^t f(s)g(s)\,ds$. Then $f(t) \leq h(t)$. We observe that $h(t) > 0$ and $h'(t) = f(t)g(t) \leq h(t)g(t)$ so that

$$\frac{h'(t)}{h(t)} \leq g(t).$$

Integrating this inequality yields $h(t) \leq C \exp\left(\int_a^t g(s)\,ds\right)$. Since $f(t) \leq h(t)$, the result follows.

If $C = 0$, use the result for $C_\varepsilon = \varepsilon$ and take limits to get h(t)=0 and hence $f(t) = 0$. $\square$

Theorem 5. *Let $\Lambda \subset \mathbb{R}^k$ and $U \subset \mathbb{R}^N$ be open. Let $X \colon U \times \Lambda \to \mathbb{R}^N$ be Lipschitz continuous on U uniformly in the variable from Λ:*

$$\|X(x,\lambda) - X(y,\lambda)\| \leq L\|x-y\|, \qquad \textit{for all } x,y \in U, \lambda \in \Lambda.$$

Fix a point $x_0 \in U$. Choose $r > 0$ such that $B(x_0,2r) \subset U$. Then there exists an $\varepsilon > 0$ and a continuous function

$$F \colon [-\varepsilon,\varepsilon] \times B(x_0,r) \times \Lambda \to B(x_0,2r)$$

such that $\frac{d}{dt}F(t,x,\lambda) = X(F(t,x,\lambda))$ and $F(0,x,\lambda) = x$ for all $x \in B(x_0,r)$, $t \in [-\varepsilon,\varepsilon]$ and $\lambda \in \Lambda$.

In fact, F is Lipschitz in x uniformly in the variables (t,λ).

Proof. We shall only highlight the arguments as the details are as in the proof of Proposition 1. We shall not write the parameter variables explicitly in what follows.

For $x \in B(x_0,r)$, consider the integral equation

$$x(t) = x + \int_0^t X(x(s))\,ds.$$

We take $\varepsilon < \min\{1/L, r/(2M)\}$. As earlier, start with $x_0 = x$ and define $x_n(t) = x + \int_0^t X(x_{n-1}(s))\, ds$. It is easily seen by induction that $x_n(s) \in B(x_0, r)$. Then x_n converges to a function $F(s,x) := \gamma_x(s)$ uniformly on $[-\varepsilon, \varepsilon]$.

To show the continuity of F, let $f(t) := \| F(t,x) - F(t,y) \|$ for $x, y \in B(x_0, r)$. We have

$$\begin{aligned} f(t) &= \left\| \int_0^t \left[X(F(s,x)) - X(F(s,y)) \right] ds + (x-y) \right\| \\ &\leq \| x - y \| + L \int_0^t f(s)\, ds \\ &\leq e^{L \,(\mathrm{mod}\, t)} \| x - y \|, \end{aligned}$$

by Gronwall's inequality. Note that this shows that the solution F is Lipschitz in the x-variable. The joint continuity follows from the observation and the fact that F is C^1 in t:

$$\begin{aligned} \| F(s,x) - F(t,y) \| &\leq \| F(s,x) - F(s,y) \| + \| F(s,y) - F(t,y) \| \\ &\leq e^{L \,(\mathrm{mod}\, s)} \| x - y \| + \| F(s,y) - F(t,y) \|. \end{aligned}$$

□

Theorem 6. *Let $X \colon U \to \mathbb{R}^n$ be a C^k vector field. Let $x_0 \in U$ be fixed. Then the function F of Theorem 5 is C^k on $(-\varepsilon, \varepsilon) \times B(x_0, r/4)$.*

Proof. Let $x \in B(x_0, r/4)$. Choose $h \in \mathbb{R}^n$ such that $\| h \| < r/4$ so that $x + h \in B(x_0, r)$. Let $F(t,x) := \gamma_x(t)$ and $F(t, x+h) := \gamma_{x+h}(t)$ be the unique solutions of the IVP with initial values x and $x+h$ respectively. We recall that F is Lipschitz in x-variable uniformly in t:

$$\| F(t, x+h) - F(t,x) \| \leq \| h \| e^{L\varepsilon}. \tag{4}$$

We now define ψ to be the matrix valued solution of the IVP:

$$\psi' = DX(F(t,x)) \circ \psi \text{ with } \psi(0) = I, \text{ Identity}.$$

Note that such a solution $\psi(t,x)$ exists, say, in $[-a, a]$ by Exer. 2 and Exer. 3.

We claim that $\frac{\partial}{\partial x} F(t,x) = \psi(t,x)$. Let $M_1 := \max\{ \| DX(x) \| : x \in B[0,r] \}$. We have

$$F(t, x+h) - F(t,x) - \psi(t,x) h \tag{5}$$

$$\begin{aligned} &= \int_0^t \left[X(F(s,x+h)) - X(F(s,x)) - DX(F(s,x)) \circ \psi(s,x) \cdot h \right] \\ &= \int_0^t DX(F(s,x)) \left[F(s,x+h) - F(s,x) - \psi(s,x) h \right] \\ &\quad + \int_0^t \left[X(F(s,x+h)) - X(F(s,x)) - DX(F(s,x)) \left(F(s,x+h) - F(s,x) \right) \right] ds. \end{aligned} \tag{6}$$

Let $f(t) := \| F(t,x+h) - F(t,x) - \psi(t,x) h \|$. The integrand in the first integral is dominated by

$$\begin{aligned} &\| DX(F(s,x)) \left[F(s,x+h) - F(s,x) - \psi(s,x) h \right] \| \\ &\leq \sup \| DX(F(s,x)) \| \, \| F(s,x+h) - F(s,x) - \psi(s,x) h \| \\ &\leq M_1 f(t). \end{aligned} \tag{7}$$

Since X is differentiable, given $\eta > 0$, there exists $\delta > 0$ such that if $\| h \| < \delta$, then

$$\| X(F(s, x + h)) - X(F(s, x)) - DX(F(s, x))\,(F(s, x + h) - F(s, x)) \| \\ < \eta \, \| F(s, x + h) - F(s, x) \| \,, \tag{8}$$

for $s \in [-\varepsilon, \varepsilon]$.

It follows from Equations 6, 7, 8 and 4, that

$$f(t) \leq M_1 \int_0^t f(s)\, ds + \eta \, \| h \| \, \varepsilon e^{L\varepsilon}, \tag{9}$$

for $t \in [-\varepsilon, \varepsilon]$, $x \in B(x_0, r/4)$ and $\| h \| < \min\{\delta, r/4\}$.

By Gronwall's inequality, it follows that for each $\eta > 0$,

$$f(t) \leq \eta \, \| h \| \, \varepsilon e^{L\varepsilon} e^{M_1 a}, \qquad t \in [-\varepsilon, \varepsilon],\ x \in B(x_0, r/4) \ \&\ \ \| h \| < \min\{\delta, r/4\}.$$

The claim is thus established. It follows from this that F is C^1 in x and C^2 in t, if X is C^1.

We prove that F is C^k in x-variable and C^{k+1} in the t-variable by induction. Since,

$$\frac{\partial}{\partial t} F(t, x) = X(F(t, x)),$$

we deduce that

$$\begin{aligned} \frac{\partial}{\partial t}\frac{\partial}{\partial t} F(t, x) &= DX(F(t, x))X(F(t, x)) \\ \frac{\partial}{\partial t}\frac{\partial}{\partial x} F(t, x) &= DX(F(t, x))\frac{\partial}{\partial x} F(t, x). \end{aligned}$$

The first equation shows that F is C^{k+1} in the t-variable while the second shows that F is C^k in the x-variable. □