Exercises in ODE

S. Kumaresan
School of Math. and Stat.
University of Hyderabad
Hyderabad 500046
kumaresa@gmail.com

1. The geometric interpretation of a vector field is to attach to each point $p \in U$, the vector $F(p)$ as a directed line segment emanating from p. A typical physical model for this concept is the velocity vector $F(p)$ in the domain of a (time-independent) fluid flow.

2. Let $f\colon [a,b] \to \mathbb{R}^n$ be continuous. If we write $f(t) = (f_1(t), \dots, f_n(t))$, we define
$$\int_a^b f(t)\,dt = \left(\int_a^b f_1(t)\,dt, \dots, \int_a^b f_n(t)\,dt\right).$$
The following observation is very useful and may be taken as the defining equation for $\int_a^b f(t)\,dt$:
$$\left\langle \int_a^b f(t)\,dt, v \right\rangle = \int_a^b \langle f(t), v\rangle\, dt, \text{ for all } v \in \mathbb{R}^n. \tag{1}$$

3. We have the fundamental theorem of calculus: $g(x) := \int_a^x f(t)\,dt$ is differentiable with derivative $g'(x) = f(x)$ for $x \in [a,b]$.

4. In particular, if $f\colon [a,b] \to \mathbb{R}^n$ is a continuously differentiable function, then $\int_a^b f'(t)\,dt = f(b) - f(a)$.

5. We have the following fundamental inequality:
$$\left\| \int_a^b f(t)\,dt \right\| \le \int_a^b \| f(t) \|\, dt.$$

6. Let $f, f_n\colon [a,b] \to \mathbb{R}^N$ be a sequence of continuous functions. Write $f_n = (f_{n1}, \dots, f_{nN})$ and $f = (f_1, \dots, f_N)$. Then $f_n \to f$ uniformly on $[a,b]$ iff $f_{nk} \to f_k$ uniformly on $[a,b]$ as $n \to \infty$.

7. We have the analogue of Weierstrass M-test. With the notation of the last item, assume that $\| f_n \| \le M_n$ for all n. Then the sequence $s_n := \sum_{j=1}^n f_j$ converges uniformly to a function $s\colon [a,b] \to \mathbb{R}^N$. We say that the series $\sum_{n=1}^\infty f_n$ is uniformly convergent on $[a,b]$ to the function s.

8. If $f_n \to f$ uniformly, then $\int_a^b f_n \to \int_a^b$.

9. Let X and Y be metric spaces. A function $f: X \to Y$ is said to be Lipschitz if there exists $L > 0$ such that $d(f(x_1), f(x_2)) \leq Ld(x_1, x_2)$ for all $x_1, x_2 \in X$. The constant L is called a Lipschitz constant of f. Note that if $L' > L$, then L' is also a Lipschitz constant of f and that any Lipschitz function is uniformly continuous.

10. Two most important examples of Lipschitz functions are (i) a linear map $A: \mathbb{R}^m \to \mathbb{R}^n$ and (ii) a differentiable map $f: U \subset \mathbb{R}^m \to \mathbb{R}^n$ with

11. f is said to be locally Lipschitz, if for each $x \in X$, there exists $r_x > 0$ such that the restriction of f to $B(x, r_x)$ is Lipschitz.

12. The most important examples of locally Lipschitz functions are: C^1 functions with bounded derivative.

Ex. 1. Show that $f(t) = t^2$ is Lipschitz on any closed and bounded subset of $\mathbb{R}$. Is it Lipschitz on $\mathbb{R}$?

Ex. 2. Show that $f(x) = 1/x$ is Lipschitz on any interval of the form $[\delta, \infty)$ where $\delta > 0$. Is it Lipschitz on $(0, \infty)$?

Ex. 3. Let $f(x,t) = x^2 e^{-t^2} \sin t$ on $R := \{(x,t) : 0 \leq x \leq 2, t \in \mathbb{R}\}$. Show that f is Lipschitz on R.

Ex. 4. Show that any linear map $A: \mathbb{R}^m \to \mathbb{R}^n$ is Lipschitz.

Ex. 5. Show that a differentiable function on a convex open set with bounded derivatives is Lipschitz. (Here the norm on any linear map A is defined as $\| A \| := \sup\{\| Ax \| : \| x \| \leq 1\}$.)

Ex. 6. Apply the method of successive approximation to the following initial value problems:

(a) $x'(t) = x(t)$ and $x(0) = 1$.
(b) $x'(t) = x + t$ and $x(0) = 0$.
(c) $x'(t) = 4tx(t)$ and $x(0) = 3$.
(d) $x'(t) = Ax(t)$ with $x(0) = b$ where A is an $n \times n$ real matrix and $b \in \mathbb{R}^n$.
(e) Data as in (d) where $n = 2$ and $A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ and $x(0) = e_1 + e_2$.
(f) Data as in (d) where $A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ and $x(0) = e_1$.

Ex. 7. Show that $f(t) = t^{2/3}$ is not Lipschitz on $\mathbb{R}$. Exhibit two distinct solutions of the IV problem: $x'(t) = 3x^{2/3}$ with $x(0) = 0$.

Ex. 8. Show that the IV problem $x' = x^3$ and $x(0) = 2$ has a solution on an interval of the form $(-\infty, b)$ for some $b \in \mathbb{R}$. What is the behaviour of $x(t)$ as $t \to b_-$?

Ex. 9. Find the integral curves of the following vector fields on the open sets $U \subset \mathbb{R}^2$:

(a) $f(x,y) = e_1$ where $e_1 = (1,0) \in \mathbb{R}^2$ on $U = \mathbb{R}^2$.
(b) $f(x,y) = v$, a fixed vector $v \in \mathbb{R}^2$ on $\mathbb{R}^2$.
(c) $f(x,y) = e_2 = (0,1)$ on $U = \mathbb{R}^2 \setminus \{0\}$.
(d) $f(x,y) = (x,y)$ on $U = \mathbb{R}^2$.
(e) $f(x,y) = (-y,x)$ on $U = \mathbb{R}^2$.
(f) $f(x,y) = (x,-y)$ on $U = \mathbb{R}^2$.

Ex. 10. Extend the basic theorem of ODE to following situation: Let Λ be any set and let $U \subset \mathbb{R}^n$ be open. Let $f\colon U \times \Lambda \to R^n$ be uniformly Lipschitz in the first variable: there exists $L > 0$ such that $\|f(x_1, \lambda) - f(x_2, \lambda)\| \leq L\|x_1 - x_2\|$ for all $\lambda \in \Lambda$. Thus the vector field f now depends on a parameter "λ". For instance, the vector field may depend on time!

Ex. 11. Solve the IV problem $x' = tx + 2t - t^3$ with $x(0) = 0$.