Pancake Problems

S. Kumaresan
School of Math. and Stat.
University of Hyderabad
Hyderabad 500046
kumaresa@gmail.com

The pancake problem can be roughly stated as follows: Suppose that two pancakes of arbitrary size lie on a plate. Is it possible to cut each of them exactly into two equal portions? If, for example, each pancake is like a disk, then the line through their centres will provide such a cut. If the pancakes are of irregular shape, then the problem becomes more difficult. The precise mathematical formulation is stated as

Theorem 1. *If A and B are two bounded regions in the same plane, the there is a line in the plane which divides each region into two regions of half area.*

Recall that a region in $\mathbb{R}^2$ is a connected open subset. Note that the theorem does not rule out the possibility that the regions may overlap.

The topological tools we decisively use here are the intermediate value theorem and the following lemma which is an immediate consequence of the intermediate value theorem.

Theorem 2 (Borsuk)**.** *Let $f\colon S^1 \to \mathbb{R}$ be continuous. Then there exists a point $x \in S^1$ such that $f(x) = f(-x)$.*

Proof. Suppose that $f(x) \neq f(-x)$ for all $x \in S^1$. We set $h\colon S^1 \to \mathbb{R}$ be the map $h(x) = f(x) - f(-x)$. We let $g\colon [0,1] \to S^1$ be the continuous map $g(t) = e^{\pi i t}$. Now, $h \circ g(0) = h(1) = f(1) - f(-1)$ and $h \circ g(1) = h(-1) = f(-1) - f(1) = -h \circ g(0)$. Therefore by intermediate value theorem, there exists a point $t \in [0,1]$ such that $h(0) = 0$, i.e., $f(-x) = f(x)$. □

The physical interpretation of this result is as follows: At any given instant of time and a given a great circle on the surface of the earth, there is a pair of antipodal points with the same temperature.

The crucial idea behind the proof of the Pancake theorem is the following observation. Let us start with a single region and a point $e^{it} \in S^1$. As we move the line perpendicular to this point from $-\infty$ to ∞ the line begins to divide the region into two parts. If we denote by $h(t)$ the area of portion of the region which is on the negative side of the direction, then $h(t)$ increases from 0 to the full area as t increases from $-\infty$ to ∞. If h is continuous, then, by intermediate value theorem there is a t_0 such that $h(t_0)$ is exactly half the area. The lemma below is a precise version of this statement.

Lemma 3. *Let $A \subset B(0,1)$ be a bounded region in $\mathbb{R}^2$. For any $x \in S^1$, we let x' stand for the antipode, the point on the circle diametrically opposite to x. Let D_x denote the diameter joining x and x'. For any $x \in S^1$, the family of all lines perpendicular to D_x contains one and only one line $L(A,x)$ which divides in half by area.*

The next lemma says that this point of intersection depends continuously on $x \in S^1$.

Lemma 4. *Let the notation be as in Thm. 1 and Lemma 3. Let x_A denote the point of intersection of D_x and $L(x,A)$. On the line D_x we have a natural coordinate g_x given by $g_x(y) = d(y,x') - 1$. Let $\varphi_A(x) := g_x(x_A)$. Then $\varphi_A \colon S^1 \to \mathbb{R}$ is continuous.*

Assuming these two lemmas, we shall prove the theorem.

Proof. (of Thm. 1) Since A and B are bounded, there is an R such that $A \subset B(0,R)$ and $B \subset B(0,R)$. We shall assume without loss of generality that $R = 1$. By Lemma 3 for B and $x \in S^1$¡ we get a unique line, say, $L(x,B)$. Let x_A (resp. x_B) denote the points of intersection of D_x and $L(x,A)$ (resp. $L(x,B)$). On the line D_x we have a natural coordinate g_x given by $g_x(y) = d(y,x') - 1$. Note that $g_x(x') = -1$, $g_x(x) = 1$ and $g_x(0) = 0$. We let $\varphi(x) := \varphi_A(x) - \varphi_B(x) \equiv g_x(x_A) - g_x(x_B)$. By Lemma 4, φ is continuous.

We observe that $\varphi(x') = -\varphi(x)$ for any $x \in S^1$. This follows from the following observations: $D_{x'} = D_x$ so that $L(A,x) = L(A,x')$ and $L(B,x') = L(B,x)$. However, $g_{x'}(y) - g_x(y)$ so that $\varphi(x') = g_{x'}(x_A) - g_{x'}(x_B) = -g_x(x_A) + g_x(x_B) = -\varphi(x)$. Now, by Thm. 2, there exists a point $x \in S^1$ such that $\varphi(x) = \varphi(x')$. It follows that $\varphi(x) = 0$ or $x_A = x_B$. Then, $L(A,x) = L(B,x)$ divides both A and B in half by area. □

Ex. 5. Let one of the regions be a regular $2n$-gon and the other a regular $2m$-gon. Can you describe the line which would bisect the regions into parts of equal area?

It now remains to give rigorous proofs of the two lemmas.

In this section we use some knowledge of fundamental groups and covering spaces to prove the following result.

Theorem 6. *There is no map $f \colon S^n \to S^1$ such that $f(-x) = -f(x)$ for $x \in S^n$ for $n \geq 2$.*

Proof. If f is such a map, then f induces a map $g \colon \mathbb{P}^n \to \mathbb{P}^1$. Recall that $\pi_n(\mathbb{P}^n) \simeq \mathbb{Z}_2$ for $n \geq 2$ while $\pi_1(\mathbb{P}^1) \simeq \mathbb{Z}$ as $\mathbb{P}^1$ is homeomorphic to S^1. Now consider a commutative diagram. The map $g_* \colon \pi_1(\mathbb{P}^n) \to \pi_1(\mathbb{P}^1)$ is the trivial homomorphism. Hence by the fundamental lemma on the lifting of maps, we see that g lifts to a map $h \colon \mathbb{P}^n \to S^1$ so that $p_1 \circ h = g$. Consider the maps $h \circ p_n, f \colon S^n \to S^1$. We claim that they are both lifts of the map $g \circ p_n \colon S^n \to \mathbb{P}^1$. For, $p_1 \circ h \circ p_n = g \circ p_n$ and $p_1 \circ f = g \circ p_n$. (Write down a commutative diagram!) Also, the maps $h \circ p_n$ and f agree at the base point. Hence by the uniqueness of the lifts, they are the same: $h \circ p_n = f$. But this is impossible, as $f(-x) \neq f(x)$ for any $x \in S^n$ whereas $h \circ p_n(-x) = h \circ p_n(x)$ for every $x \in S^n$! This contradiction shows that such an f does not exist. □

Theorem 7. *If $f \colon S^2 \to \mathbb{R}^2$ is continuous, then there is an $x \in S^1$ with $f(x) = f(-x)$.*

Proof. If the result were false, then the map $x \mapsto f(x) - f(-x)$ is never zero on S^2 so that the map $g(x) := \frac{f(x)-f(-x)}{\|f(x)-f(-x)\|}$ is continuous. Also, we have $g(-x) = -g(x)$, contradicting Thm. 6 $\square$

Ex. 8. There is no homeomorphic copy of S^2 in $\mathbb{R}^2$.

This exercise says that there could be no homeomorphic copy of the earth on the paper of an atlas!

Theorem 9 (Lusternik-Schnirelmann Theorem)**.** *If $S^2 = \cup_{i=1}^3 F_i$ is the union of three closed subsets then one of the f_i's contains a pair of antipodal points.*

Proof. Let x' denote the antipodal point $-x$ of a given point and A' the set of antipodal points of the elements of a set A. If $F_i \cap F_i' \neq \emptyset$ for $i = 1, 2$ we have nothing prove. So we assume that $F_i \cap F_i' = \emptyset$ for $i = 1, 2$. Then, by Urysohn's lemma, there exist continuous functions g_i such that $g_i \colon S^1 \to [0, 1]$ and $g_i = 0$ on A_i and $g_1 = 1$ on A_i'. Consider the map $f \colon S^2 \to \mathbb{R}^2$ given by $f(x) := (g_1(x), f_2(x))$ for $x \in S^2$. By Thm. 7, there is an $x \in S^2$ such that $f(x) = f(-x)$. We claim that this $x \notin F_i$ for $i = 1, 2$. For, if $x \in F_1$, say, then $f(x) = (g_1(x), g_2(x)) = (g_1(x'), g_2(x')) = f(x')$ would imply that $0 = g_1(x) = g_1(x') = 1$—a contradiction. Hence we conclude that $x \in F_3$. A similar reasoning shows that $x' \in F_3$. $\square$

Incomplete!