Sign of a Permutation

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Theorem 1. *If n is an integer greater than 1 and S_n is the symmetric group on the n symbols $\{1, 2, \ldots, n\}$ then the map $\varepsilon \colon S_n \to \mathbb{Q}^\times$ given by*

$$\sigma \mapsto \prod_{1 \le i < j \le n} \frac{\sigma j - \sigma i}{j - i}$$

is a group homomorphism. Its image is $\{\pm 1\}$.

Proof. Let us show that ε is a group homomorphism. Let X be the collection of two element subsets of $\{1, 2, \ldots, n\}$. For each $\sigma \in S_n$ and every $\{i, j\} \in X$ define

$$\sigma'(\{i, j\}) := \frac{\sigma j - \sigma i}{j - i} \ ;$$

the right hand side is symmetric in i and j and hence σ' is well-defined. We have

$$\varepsilon(\sigma) = \prod_{x \in X} \sigma'(x). \tag{1}$$

S_n acts in a natural way on X. For $\sigma, \tau \in S_n$ and $x \in X$, we show that

$$(\sigma\tau)'(x) = \sigma'(\tau x)\ \tau'(x). \tag{2}$$

Let $x := \{i, j\}$. We compute

$$\begin{aligned}
(\sigma\tau)'(x) &= (\sigma\tau)'(\{i, j\}) \\
&= \frac{\sigma\tau j - \sigma\tau i}{j - i} \\
&= \frac{\sigma\tau j - \sigma\tau i}{\tau j - \tau i} \ \frac{\tau j - \tau i}{j - i} \\
&= \sigma'(\tau\{i, j\})\ \tau'(\{i, j\}) \\
&= \sigma'(\tau x)\ \tau'(x).
\end{aligned}$$

In view of Eq. 1 and Eq. 2 we obtain

$$\begin{aligned}
\varepsilon(\sigma\tau) &= \prod_{x\in X} (\sigma\tau)'(x) \\
&= \prod_{x\in X} \big(\sigma'(\tau x)\ \tau'(x)\big) \\
&= \prod_{x\in X} \sigma'(\tau x) \quad \prod_{x\in X} \tau'(x) \\
&= \varepsilon(\sigma)\ \varepsilon(\tau).
\end{aligned}$$

If τ is the transposition (1,2) then $\varepsilon(\tau) = -1$. $\square$