Partial Order, Total Order and Well-ordering

S. Kumaresan
School of Math. and Stat.
University of Hyderabad
Hyderabad 500046
kumaresa@gmail.com

1. Definition of a relation R between two sets X and Y as a subset of the product $X \times Y$. Definition of a relation on a set X is a relation between X and X. Given $(x, y) \in R$, we write xRy. Later we shall write $x \leq y$ in place of xRy or $(x, y) \in R$.

2. Given a relation R between X and Y, the inverse relarion R^{-1} between Y and X is given by $R^{-1} := \{(y, x) \in Y \times X : (x, y) \in R\}$.

3. Definition when a relation is reflexive, symmetric, antisymmetric and transitive.

4. Definition of partial order, total order and equivalence. A set X with a partial order $\leq$ (or the pair $(X, \leq)$) is called a partially ordered set (or a poset for brevity).

5. Typical examples.

   (a) Standard $\leq$ relation on $\mathbb{Z}$, $\mathbb{Q}$ and $\mathbb{R}$.

   (b) In $P(X)$, $A \leq B$ iff $A \subseteq B$. This is a partial order which is not a total order. (Perhaps, the most important example!) We say that this order is defined by the inclusion.

   (c) In $\mathbb{N}$, we define $a \preceq b$ iff a divides b. This is a partial order which is not a total order.

   (d) The relation $R := \{(x, x)\}$ is an equivalence as well as a partial order. This is nothing other than equality relation.

   (e) If a relation R on X is a partial order, then so is R^{-1}.

   (f) In $\mathbb{R}^2$, we define $(x_1, y_1) \leq (x_2, y_2)$ iff either $x_1 < x_2$ (and no requirement on y_1 and y_2) or $x_1 = x_2$ and $y_1 \leq y_2$. This is a total order, called dictionary or lexicographic order.

   (g) $\mathbb{C}$ is a totally ordered set. Compare this with what you learnt in complex analysis. Most students have problem here.

6. Given relations R on X and S on Y, we have a relation T on $X \times Y$ by setting $(x_1, y_1)T(x_2, y_2)$ if $x_1 R x_2$ and if $x_1 = x_2$, then $y_1 S y_2$. If we agree to denote all the relations by $\leq$, then the realtion on $X \times Y$ is defined as follows:

$$(x_1, y_1) \leq (x_2, y_2) \text{ iff } x_1 \leq x_2 \text{ and if } x_1 = x_2 \text{ then } y_1 \leq y_2.$$

This is known as lexicograhic or dictionary relation on $X \times Y$. If the relations on X and Y are partial orders (respectively, total orders), the lexicographic relation is a partial order (respectively total order) on $X \times Y$.

7. Consider the standard total order $\leq$ on $\mathbb{R}$. For $(x_1, y_1), (x_2, y_2) \in \mathbb{R}^2$, define $(x_1, y_1) \leq (x_2, y_2)$ iff $x_1 \leq x_2$ and $y_1 \leq y_2$. Show that the new relation on $\mathbb{R}^2$ is a partial order which is not a total order. (Can you generalise this result?)

8. Restriction of a partial order to a subset. Chain = totally ordered set.

9. Consider the subset $A := \{2^n : n \in \mathbb{N}\}$ in the partially ordered set of Item 5c. Then A is a chain.

10. A chain need not be a countable set. For example, consider $X = [0, 1]$ and $P(X)$ with inclusion as the partial order. Then the family $\{[t, 1] : 0 < t \leq 1\}$ is a chain in $P(X)$.

11. Definition of an upper bound, lower bound, l.u.b., and g.l.b. Examples in $\mathbb{R}$ and $P(\mathbb{R})$.

12. Definition of minimum, maximum, minimal and maximal elements. Maximum and minimum are unique.

13. Any maximum (in fact, the maximum) is a maximal element but the converse is not true. Analogue for minimum and minimal elements.

14. In a totally ordered set, a maximal (respectively, minimal) element is a maximum (respectively, a minimum).

15. Any finite poset has a maximal element. How about the existence of minimal elements? (A finite poset may not have a maximum! See Item 19.)

16. Any finite totally ordered set has a maximum. How about minimum?

17. Discuss the minimal and maximal elements in (1) $P(\mathbb{R})$, (2) $P(\mathbb{R}) \setminus \{\emptyset\}$, (3) $P(\mathbb{R}) \setminus \{\mathbb{R}\}$, (4) $P(\mathbb{R}) \setminus \{\emptyset, \mathbb{R}\}$ all being ordered by inclusion.

18. Let X denote the set of $\{\{0\}, \{x\}, \{y\}, \{z\}, \{x, y\}, \{x, z\}, \{y, z\}, \mathbb{R}^3$ where $\{x\}$ denotes the x-axis, $\{x, y\}$ denote the xy-plane etc. Define a partial order $\leq$ on X by inclusion. Give a pictorial representation of this partial ordered set. (The picture is called the Hasse diagram of the poset.) Discuss the maximum, minimum elements in X. If $Y := X \setminus \{\{0\}, \mathbb{R}\}$, what are the maximum, minimum, maximal and minimal elements (if they exist)?

19. Consider $X := \{2^m 3^n : m, n \in \mathbb{Z}_+,\ 1 \leq m + n \leq 4\}$ with the partial order as in Item 5c. Find the maximum, minimum, maximal and minimal elements. Give a graphical representation of this poset.

20. Let $R \subset \mathbb{R} \times \mathbb{R}$ be the relation that corresponds to the standard order $\leq$ on $\mathbb{R}$. Draw the picture of the subset R.

21. Discuss the minimal and maximal elements in (1) $\mathcal{H}_1$, the set of all nontrivial proper subgroups, (2) $\mathcal{H}_2$, the set of all proper subgroups and $\mathcal{H}_3$, and (3) the set of all subgroups of the group $(\mathbb{Z}, +)$.

22. Are there partially ordered sets in which every maximal element is a minimal element and vice-versa?

23. If X is a partially ordered set in which the maximum is also a minimum what can you conclude about X?

24. Define the LUB and GLB of a subset in a poset. Find the LUB and GLB of $m\mathbb{Z}$ and $n\mathbb{Z}$ in the partially ordered set $\mathcal{H}$ of all subgroups of $\mathbb{Z}$.

25. Let X denote the set of all non-empty subsets of $\mathbb{R}$ which miss at least one integer, ordered by inclusion. Then X has countably infinite maximal elements while it has uncountably many minimal elements.

26. Let $\mathcal{S}$ denote the set of nontrivial proper vector subspaces V of $\mathbb{R}^n$ (that is, $V \neq (0)$ and $V \neq \mathbb{R}^n$) where $n \geq 2$. For $V, W \in \mathcal{S}$, we define $V \leq W$ if $V \subseteq W$. Characterize the maximal and the minimal elements of $\mathcal{S}$.

27. Let V be a (not necessarily finite dimensional) vector space over $\mathbb{R}$. We say that $S \subset V$ is linearly independent if every finite subset S is linearly independent. Let $\mathcal{S}$ be the set of all linearly independent subsets of V ordered by inclusion. Let B be a maximal element of $\mathcal{S}$. Show that the linear span of B is V.

28. Let V be a nonzero (real) vector space. Let $\mathcal{S}$ denote the set of all linearly independent subset of V ordered by inclusion. Then the maximal elements of $\mathcal{S}$ are nothing other than the bases of V.

29. Let R be a commutative ring with identity 1. Let I be a maximal element in the set of all proper ideals of R ordered by inclusion. Show that for any $x \notin I$, there exist $r \in R$ and $z \in I$ such that $rx + z = 1$.

30. Consider the set $X := \{(f, S)\}$ where S is a subinterval of $[0, 1]$ such that $(1/2, 3/4) \subset S$ and $f : S \to \mathbb{R}$ is continuous such that $f(x) = 1/x$ for $x \in [1/2, 3/4]$. We say $(f, S) \leq (g, T)$ if $S \subseteq T$ and $g(x) = f(x)$ for $x \in S$. Write down an infinite set of maximal elements of X. 3/4) by $g(x) = 1/x$ for $x \geq 1/2$ and $g(x) = \frac{1-2t}{x-t}$. Then $\lim_{x \to t_+} g(x) = \infty$, $g(1/2) = 2$. Thus g is continuous but has no extension.

31. Intervals in a totally ordered set. Give a geometric and explicit description of the intervals $[(0, 0), (0, 1)]$ and $[(0, 0), (1, 0)]$ and $[(a, b), (c, d)]$ in $\mathbb{R}^2$ with dictionary order.

32. **Zorn's Lemma.** Let X be a partially ordered set. Assume further that the partial order is such that every chain in X has an upper bound. Then there exists a maximal element in X.

33. The assumption on the partial order in Zorn's lemma is important. Consider $\mathcal{S} := \{S_t := (0, t) : 0 < t < 1\}$ ordered by inclusion. The order is total and hence $\mathcal{S}$ is a chain. It is easy to see that the chain has no upper bound in $\mathcal{S}$ and that $\mathcal{S}$ has no maximal element.

34. Show that nay vector space over a field has a basis. *Hint.* Items 27, 28 and 32 are relevant.

35. **Axiom of Choice.** Let $\{X_i : i \in I\}$ be a nonempty family of nonempty sets. Then there exists a set A which has exactly one element from each X_i, $i \in I$.

    This can be rephrased as follows. The cartesian product $\prod_{i \in I} X_i$ is not empty!

36. Let $(X, \leq)$ be a poset. We say that the order $\leq$ is a well-ordering on X if it is (i) a total order and (ii) every nonempty subset of X has a minimum.

37. **Well-ordering Principle.** On any set, there exists a well-ordering.

38. It can be proved that the axiom of choice, Zorn's lemma and the well-ordering principle are equivalent. Perhaps the most used of these is Zorn's lemma followed by the axiom of choice.

39. Zorn's lemma is used in the following results (in a typical M.Sc. course):

    (a) Existence of a basis for any vector space.

    (b) Existence of a maximal ideal in a ring with 1.

    (c) Existence of algebraic closures of fields.

    (d) Tychonoff's theorem which asserts that the product of a family of compact spaces is compact in the product topology.

    (e) Hahn-Banach extension theorem in functional analysis.

40. The axiom of choice is used in the construction of a non-measurable set in the theory of Lebesgue measure.