Existence of Smooth Functions and Partition of Unity

S. Kumaresan
School of Math. and Stat.
University of Hyderabad
Hyderabad 500046
kumaresa@gmail.com

1 Smooth Functions with Compact Support

Definition 1. Let $f\colon X \to \mathbb{C}$ be a function from a topological space X to $\mathbb{C}$. Then *support* of f in X is the set

$$\operatorname{Supp} f := \overline{\{x \in X : f(x) \neq 0\}}.$$

If $X := U \subset \mathbb{R}^n$ is an open set, then the support of f in U is denoted by $\operatorname{Supp}_U(f)$ for more emphasis.

Ex. 2. Let $f\colon [a,b] \to \mathbb{R}$ be continuous and C^1 on (a,b). Assume that $\lim_{x\to a+} f'(x) = l$ and $\lim_{x\to b-} f'(x) = m$ exist. Show that f is C^1 on $[a,b]$.

Ex. 3. Let $f\colon U \subset \mathbb{R}^n \to \mathbb{R}$ be continuous on U and C^1 on $U \setminus \{a\}$ for $a \in U$. Assume that $\ell_i := \lim_{x\to a} D_i f(x)$ exists for $1 \leq i \leq n$. Prove that $D_i f(a) = \ell_i$ and that f is C^1 on U.

Ex. 4. Consider $f : \mathbb{R} \to \mathbb{R}$ defined by

$$f(t) = \begin{cases} 0 & \text{for } t \leq 0 \\ \exp(-1/t) & \text{for } t > 0. \end{cases}$$

f is differentiable on $\mathbb{R}^n \setminus \{0\}$.

(a) Observe that $e^x > \frac{x^k}{k!}$ for $k \in \mathbb{N}$.

(b) Prove that $f(x) < k! x^k$ for $k \in \mathbb{N}$ and hence conclude that f is continuous at $x = 0$.

(c) Prove by induction that $f^{(k)}(x) = p_k(x^{-1}) f(x)$ for some polynomial of degree less than or equal to $k+1$ (for $x \neq 0$). Note that

$$\begin{aligned} \left| \left[f^{(k)}(x) - f^{(k)}(0) \right] x \right| &= \left\| f(x) x^{-1} p_k(x^{-1}) \right\| \\ &\leq n! x^{n-k}. \end{aligned}$$

Conclude that $f^{(k+1)}(0)$ exists and hence f is infinitely differentiable on all of $\mathbb{R}$.

Ex. 5. Carry out a similar analysis to conclude that $f : \mathbb{R} \to \mathbb{R}$ defined by

$$f(t) = \begin{cases} e^{-\frac{1}{t^2}} & t > 0 \\ 0 & t \leq 0 \end{cases}$$

is infinitely differentiable.

Ex. 6. Let f be as in Ex. 5. Let $\varepsilon > 0$ be given. Define $g_\varepsilon(t) := f(t)/(f(t) + f(\varepsilon - t))$ for $t \in \mathbb{R}$. Then g_ε is differentiable, $0 \le g_\varepsilon \le 1$, $g_\varepsilon(t) = 0$ iff $t \le 0$ and $g_\varepsilon(t) = 1$ iff $t \ge \varepsilon$.

Ex. 7. For $x \in \mathbb{R}^n$ and $\varepsilon > 0$, there exists a smooth function $f\colon \mathbb{R}^n \to [0, \infty)$ such that $f^{-1}((0, \infty)) = B(x, \varepsilon)$.

Ex. 8. Let f, g be as in Ex. 6. For $r > 0$ and $x \in \mathbb{R}^n$, define $\varphi(x) := 1 - g_\varepsilon(\|x\| - r)$. Then φ is smooth and has the following properties:
(i) $0 \le \varphi \le 1$, (ii) $\varphi(x) = 1$ iff $\|x\| \le r$ and $\varphi(x) = 0$ iff $\|x\| \ge r + \varepsilon$.

Ex. 9. Let $\psi(u) = u^{-k}e^u$ for $u > 0$. Show that

$$\begin{aligned} \psi'(u) &= (u-k)u^{-k-1}e^u \\ \psi''(u) &= [u^2 - 2ku + k(k+1)]u^{-k-2}e^u. \end{aligned}$$

Show that the expression in the brackets has a minimum when $u = k$ and is positive at $u = k$. Hence $\psi''(u) > 0$ and for any u_0

$$\psi(u) \ge \psi(u_0) + \psi'(u_0)(u - u_0).$$

If $u_o > k$, then $\psi'(u_0) > 0$ and the right hand side above tends to infinity as $n \to \infty$. Hence conclude that $\psi(u) \to \infty$ as $u \to \infty$.

Ex. 10. Use the above exercise to prove that f as defined in Ex. 5 is smooth.

Ex. 11. Let $0 < a < b$. Consider the functions $f_a\colon R \to \mathbb{R}$ given by $f_a(t) = \exp(-1/(t-a))$ for $t \ge a$ and 0 otherwise. and $g_b\colon \mathbb{R} \to \mathbb{R}$ given by $g_b(t) = \exp(1/(t-b))$ for $t \le b$ and 0 otherwise. Then the product φ of these functions is a smooth function which is 0 outside the interval $[a, b]$. Set $\eta(x) := \varphi(\|x\|)$ for $x \in \mathbb{R}^n$. List the properties of η.

Ex. 12. Let φ be as in Ex. 11. Define h on $\mathbb{R}$ as follows.

$$h(x) := \left(\int_x^b \varphi(t)\, dt\right)\left(\int_a^b \varphi(t)\, dt\right)^{-1}.$$

Then h is smooth with $h(x) \le 1$ for $x \le a$ and $h(x) = 0$ if $x \ge b$. Define $\psi(x) := h(\sum_i x_i^2)$ for $x := (x_1, \ldots, x_n) \in \mathbb{R}^n$, then $\psi(x) = 1$ for $x \in B(0, a)$ and $\psi(x) = 0$ for $\|x\| \ge b$.

Ex. 13. If K is a compact set in $\mathbb{R}^n$ and U is an open set containing K then there exists a smooth function f on $\mathbb{R}^n$ which is 1 on K and 0 outside U (i.e., 0 on $\mathbb{R}^n \setminus U$).

2 Partition of Unity

Lemma 14. *Let $U \subset \mathbb{R}^n$ be open. There exist compact subsets K_j, $j \in \mathbb{N}$ such that $\cup_j K_j = U$ and such that*

$$K_j \subset \text{Interior of } K_{j+1} \subset K_{j+1} \text{for all } j \in N.$$

Proof. Define
$$K_j := \{x \in U : \|x\| \leq j \text{ and } d(x, \mathbb{R}^n \setminus U) \leq 1/j\}.$$

Alternatively, let $\{U_j\}$ be any countable cover of U such that $\overline{U}_j$ is compact and $\overline{U_j} \subset U$. Let $K_1 = \overline{U}_1$. Take $K_2 = \overline{U}_1 \cup \cdots \cup \overline{U}_r$ where r is the first integer such that $K_1 \subset U_1 \cdots \cup U_r$. If $K_m := \overline{U}_1 \cup \cdots \overline{U}_s$, then we take $K_{m+1} := \overline{U_1} \cup \cdots \cup \overline{U}_p$ where p is minimal integer so that $K_m \subset U_1 \cup \cdots \cup U_p$. $\square$

Proposition 15. *Let $U \subset \mathbb{R}^n$ be open. Let $\{U_\alpha : \alpha \in I\}$ be an open cover of U. Then there exist a sequence (x_i) in U and a sequence (ε_i) of positive reals such that*

(i) $U = \cup_{i=1}^{\infty} B(x_i, \varepsilon_i)$,

(ii) For each $i \in \mathbb{N}$, there exists an $\alpha \in I$ such that $B(x_i, 2\varepsilon_i) \subset U_\alpha$,

(iii) Each $x \in U$ has an open neighbourhood W such that $\{i : W \cup B(x_i, 2\varepsilon_i) \neq \emptyset\}$ is a finite set.

Proof. Choose K_m as in the last lemma. Let $K_0 = \emptyset = K_{-1}$. For each $m \geq m$, we define the following sets:
$$C_m := K_m \setminus \text{Int} K_{m-1}, \quad W_m := \text{Int} K_{m+1} \setminus K_{m-2}.$$
Clearly, C_m is compact, W_m is open, $C_m \subset W_m$ and $U = \cup_m C_m$.

For each $x \in C_m$, we can find $\varepsilon(x) > 0$ such that $B(x, 2\varepsilon(x)) \subset W_m$ and such that there exists an α such that $B(x, 2\varepsilon(x)) \subset U_\alpha$. Since C_m is compact, by Heine-Borel theorem, there exist a finite number of points, say, $x_{m,j}$ and positive constant $\varepsilon_{m,j}$, $1 \leq j \leq k_m$, such that

(A) $C_m \subset \cup_{j=1}^{k_m} B(x_{m,j}, \varepsilon_{m,j}$

(B) Each $B(x_{m,j}, 2\varepsilon_{m,j}) \subset W_m$ and is also contained in at least one member U_α of the original open cover.

To arrive at the result, we can reindex the countable families $(x_{m,j})$ and $(\varepsilon_{m,j})$. We now have
$$U = \cup_m C_m \subset \cup_{m=1}^{\infty} \cup_{j=1}^{k_m} B(x_{m,j}, \varepsilon_{m,j}) \subset \cup_m W_m \subset U.$$
Thus (i) is established. (ii) follows from our choice of ε's. To prove (iii), let $x \in U$. Choose m such that $x \in W_m$. Since $W_m \cap W_k = \emptyset$ whenever $k \geq m+3$, it follows that W_m can intersect $B(x_{k,j}, 2\varepsilon_{k,j})$ only when $k \leq m+2$. This establishes (iii). $\square$

Theorem 16 (Existence of Smooth Partition of Unity)**.** *Let $U \subset \mathbb{R}^n$ be open. Let $\{U_\alpha | \alpha \in I\}$ be an open cover of U. Then there exist smooth functions $f_\alpha \colon U \to [0,1]$ with the following properties:*

(i) Supp $f_\alpha \subset U_\alpha$ for each $\alpha in I$.

(ii) Each $x \in U$ has a neighbourhood W such that $\{\alpha \in I : f_\alpha \neq 0 \text{ on } W\}$ is finite.

(iii) For each $x \in U$, we have $\sum_{\alpha \in I} f_\alpha(x) = 1$.

Remark 17. The family $\{\text{Supp } f_\alpha\}$ satisfying the condition (ii) of the theorem is said to be *locally finite*. Note that the sum in (iii) is a finite sum so that it is a well-defined real valued function. It is also smooth. (Why?)

Proof. Choose (x_i), (ε_i) and $\alpha = \alpha_i$ as in Proposition 15. Let $g_i \colon \mathbb{R}^n \to [0, \infty)$ be smooth functions such that $g_i^{-1}(0, \infty) = B(x_i, \varepsilon_i)$. Condition (iii) of the Proposition ensures that $\sum_i g_i(x)$ is a finite sum on a neighbourhood of each point $x \in U$ and hence it is a smooth function, say, g on U. From condition (i) of the Proposition, it follows that $g(x) > 0$ for each $x \in U$.

Let $h_i(x) := g_i(x)/g(x)$. Then h_i are smooth, $h_i^{-1}(0,\infty) = B(x_i, \varepsilon_i)$ and $\sum_i h_i(x) = 1$ for all $x \in U$. Define $f_\alpha := \sum h_i$ where the sum is over only those i for which $\alpha_i = \alpha$. It is easy to see that f_α are smooth. To show that f_α's are as required, let $x \in \operatorname{Supp} f_\alpha$. Then by Proposition 15, x has a neighbourhood W in U satisfying condition (iii) of the Proposition. Then the restriction of f_α to W is the sum of finitely many h_i such that $\alpha_i = \alpha$. There is at least one such i such that $x \in \operatorname{Supp} h_i$. We now observe that

$$\operatorname{Supp} h_i = B[x_i, \varepsilon_i] \subset B(x_i, 2\varepsilon_i) \subset U_\alpha.$$

Thus, $x \in U_\alpha$. Condition (i) of the theorem follows. The other conditions are obvious. □

Ex. 18. Let $K \subset \mathbb{R}^n$ be a closed set. A function $f\colon K \to \mathbb{R}$ is said to be smooth iff for each x, there exists an open set U_x and a smooth function $f_x\colon U_x \to \mathbb{R}$ such that f is the restriction of f_x to $U_x \cap K$. Show that $f\colon K \to \mathbb{R}$ is smooth iff there exists an open set U and a smooth function $g\colon U \to \mathbb{R}$ such that $K \subset U$ and f is the restriction of g to K.

Lemma 19. *Let K be a closed subset of U and $U_0 \subset U$ be open containing K. Let $f\colon U \to V$ be continuous. Assume further that f is smooth on U_0. Let $\varepsilon\colon U \to (0,\infty)$ be given. Then there exists a smooth $g\colon U \to V$ such that the following hold:*
(i) $g(x) = f(x)$ for $x \in K$.
(ii) $\| g(x) - f(x) \| < \varepsilon(x)$ for $x \in U$.
(iii) the line segment $[f(x), g(x)] \subset V$ for $x \in U$.

Proof. Let $\varepsilon\colon U \to (0,\infty)$ be given. Let

$$\varepsilon'(x) = \min\{\varepsilon(x), \frac{1}{2} d(f(x), \mathbb{R}^n \setminus V)\}.$$

For $p \in U \setminus K$, using the continuity of f at p, we can choose a neighborhood $U_p \subseteq U \setminus K$ such that

$$\| f(x) - f(p) \| < \varepsilon'(p) \text{ for all } x \in U_p. \tag{1}$$

Let $\mathcal{U} = U_0 \cup \{\cup U_p\}_{p \notin K}$. Then $\mathcal{U}$ is an open cover for U. Let $g_0 \cup \{g_p\}_{p \in U\setminus K}$ be a partition of unity subordinate to the open cover $\mathcal{U}$. This means that
(i) $\operatorname{Supp} g_0 \subseteq U_0$ and $\operatorname{Supp} g_p \subseteq U_p$.
(ii) Each $x \in U$ has a neighborhood W_x on which only finitely many g's do not vanish.
(iii) $g_0(x) + \displaystyle\sum_{p \in U\setminus K} g_p(x) = 1,$ for $x \in U$. Define

$$g(x) := g_0(x) f(x) + \sum_{p \in U\setminus K} g_p(x) f(p). \tag{2}$$

Note that for $x \in K$, $g_p(x) = 0$ for all $p \in U \setminus K$. By (iii) $g_0(x) = 1$ and hence by Eqn. 2, $g(x) = f(x)$ on K.

For $x \in U$,

$$\begin{aligned}
\| f(x) - g(x) \| &= \left\| \sum_{p \in U \setminus K} g_p(x) f(x) - \sum_{p \in U \setminus K} g_p(x) f(p) \right\| \\
&= \left\| \sum_{p \in U \setminus K} g_p(x)[f(x) - f(p)] \right\| \\
&\leq \| f(x) - f(p) \| \\
&< \varepsilon'(x) \qquad\qquad \text{(by Eqn. 1)} \\
&< \frac{1}{2} d(f(x), \mathbb{R}^n \setminus V).
\end{aligned}$$

Hence the line segment $[f(x), g(x)] \subset V$. This completes the proof. □

Corollary 20. *Let $f\colon U \to V$ be a continuous map. Then there exists a smooth map $g\colon U \to V$ which is homotopic to f.* □