# Some Properties of Harmonic Functions

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**Theorem 1** (Converse of Mean Value Property-I). *Let $u \in C^2(\Omega)$ satisfies the mean value property:*

$$u(x) = \int_{S(x,r)} u(y)\, dS(y)$$

*for all $B(x,r) \subset \Omega$, then $u$ is harmonic.*

*Proof.* If $\Delta u(x) > 0$, say, for some $x \in \Omega$, by continuity we can find an $r > 0$ such that $\Delta u > 0$ on $B(x,r)$. If we set $\varphi(\rho) := \frac{1}{\omega_n} \int_{|\xi|=1} u(x+\rho\xi)\, dS(\xi)$, then $\varphi'(r) = 0$. But then

$$\varphi'(r) = \frac{d}{dr} \int_{S(x,r)} u(y)\, dy = \int_{S(x,r)} \frac{\partial u}{\partial \nu}\, dS(y) = \int_{B(x,r)} \Delta u(y) dy > 0,$$

a contradiction. □

**Theorem 2** (Converse of Mean Value Property-II). *Let $u \in C(\Omega)$ have the mean value property: for all $x \in \Omega$ and for all $r > 0$ with $B[x,r] \subset \Omega$ we have*

$$u(x) = \frac{1}{\omega_n} \int_{|\xi|=1} u(x+r\xi)\, dS(\xi).$$

*Then $u \in C^\infty(\Omega)$ and $u$ is harmonic in $\Omega$.*

*Proof.* Let $\varphi$ be a smooth radial cut-off function with support in the unit ball: $u \in C_c^\infty(B(0,1))$ with $\int \varphi = 1$ and $\varphi(x) = \varphi(|x|)$. Let $\varepsilon > 0$. We set $\Omega_\varepsilon$ to be the set of points $x \in \Omega$ such that $d(x, \partial\Omega) > \varepsilon$. Let $\varphi_\varepsilon$ be defined as usual: $\varphi_\varepsilon(x) := \varepsilon^{-n}\varphi(x/\varepsilon)$. If $x \in \Omega_\varepsilon$, the function $y \mapsto \varphi_\varepsilon(x-y)$ has support inside $\Omega$ and hence the convolution $u \star \varphi_\varepsilon$ makes sense. We have

$$\begin{aligned}
\int u(x-y)\varphi_\varepsilon(y)\, dy &= \int u(x-y)\varepsilon^{-n}\varphi(y/\varepsilon)\, dy \\
&= \int u(x-\varepsilon y)\varphi(y)\, dy \\
&= \int_0^\infty \int_{|\xi|=1} u(x-\varepsilon r\xi) dS(\xi) \varphi(r) r^{n-1}\, dr \\
&= \omega_n u(x) \int_0^\infty \varphi(r) r^{n-1}\, dr \\
&= u(x) \int \varphi(y)\, dy = u(x).
\end{aligned}$$

Thus we have $u \star \varphi_\varepsilon = u$ on $\Omega_\varepsilon$. It is well-known that $u \star \varphi_\varepsilon$ is $C^\infty$ and hence $u$ is so on $\Omega_\varepsilon$. Letting $\varepsilon \to 0$ we get the result.

To prove that $u$ is harmonic, we shall apply Gauss-Green theorem.

$$\begin{aligned}
\int_{B(x,r)} \Delta u &= \int_{S(x,r)} \frac{\partial}{\partial \nu} u dS \\
&= \frac{d}{dr} \int_{S(0,1)} u(x + r\xi) dS(\xi) \\
&= \frac{d}{dr} u(x) = 0.
\end{aligned}$$

□

**Corollary 3.** *If $(u_j)$ is a sequence of harmonic functions on $\Omega$ which converge uniformly on compact subsets of $\Omega$ to a function $u$ then $u$ is harmonic.*

**Theorem 4** (Liouville)**.** *If $u$ is a bounded harmonic function on $\mathbb{R}^n$, then $u$ is a constant.*

*Proof.* Assume that $|u(x)| \leq M$ on $\mathbb{R}^n$. We shall use the mean value property in the following form:

$$u(x) = \frac{n}{R^n \omega_n} \int_{B(x,R)} u(y)\, dy. \tag{1}$$

For any $x \in R^n$ and $R > |x|$, we have

$$\begin{aligned}
|u(x) - u(0)| &= \frac{n}{R^n \omega_n} \left| \int_{B(x,R)} u(y)\, dy - \int_{B(0,R)} u(y)\, dy \right| \\
&\leq \frac{n}{R^n \omega_n} M \left[ \int_{|y|<R, |x-y|>R} dy + \int_{|y|>R, |x-y|<R} dy \right] \\
&\leq \frac{n}{R^n \omega_n} M \int_{R-|x|<|y|<R+|x|} dy \\
&= \frac{n}{R^n} M \int_{R-|x|}^{R+|x|} r^{n-1}\, dr \\
&= \frac{M}{R^n} [(R + |x|)^n - (R - |x|)^n] \\
&\to 0 \text{ as } R \to \infty.
\end{aligned}$$

□

**Theorem 5.** *Let $u$ be a nonnegative harmonic function on $\Omega$. Then for any connected subdomain $\Omega' \subset \Omega$ with compact closure, there exists a constant $C > 0$ depending only on $\Omega'$ such that*

$$\sup_{\Omega'} u \leq C \inf_{\Omega'} u.$$

*In particular,*

$$C^{-1} u(y) \leq u(x) \leq C u(y), \text{ for all points } x, y \in \Omega'.$$

*Proof.* Let $y \in \Omega$ and $R > 0$ be such that $B(y, 4R) \subset \Omega$. Then for any two points $x_1, x_2 \in B(y, R)$ we have by the mean value theorem, (see the proof of Thm. 4 (1)),

$$u(x_1) = \frac{1}{R^n \omega_n} \int_{B(x_1,R)} u(z)\, dz \leq \frac{1}{R^n \omega_n} \int_{B(y,2R)} u(z)\, dz. \tag{2}$$

$$u(x_2) = \frac{1}{(3R)^n \omega_n} \int_{B(x_2,3R)} u(z)\, dz \geq \frac{1}{(3R)^n \omega_n} \int_{B(y,2R)} u(z)\, dz. \tag{3}$$

From these two, we obtain

$$\sup_{B(y,R)} u \leq 3^n \inf_{B(y,R)} u. \tag{4}$$

Choose $x_1, x_2 \in \overline{\Omega'}$ such that $u(x_1) = \sup_{\Omega'} u(x)$ and $u(x_2) = \inf_{\Omega'} u(x)$. Let $\gamma$ be a continuous path joining $x_1$ and $x_2$. Choose $R$ so that $4R < d(\gamma, \partial\Omega)$. By compactness, $\gamma$ can be covered by a finite number $N$ (depending only on $\Omega'$ and $\Omega$) of balls with radius $R$. Applying the estimate (4) in each of these balls and combining the resulting inequalities, we get

$$u(x_1) \leq 3^{nN} u(x_2).$$

□