Proofs of Malgrange-Ehrenpreis Theorem

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Introduction

In the first few sections, we shall look at several proofs of the local solvability of constant coefficient linear PDE

$$P(D)u = f, \quad \text{for } f \in L^2(\Omega). \tag{1}$$

In the last section we shall give proofs for the existence of fundamental solutions for $P(D)$, i.e., existence of a distribution E such that $P(D)E = \delta$, the Dirac distribution.

Notataions. We use the standard notation of PDE.

Let Ω denote an open set in $\mathbb{R}^n$. Let $C_0^\infty(\Omega)$ denote the space of smooth (C^∞) functions with compact support in Ω. For $\alpha \in \mathbb{Z}_+^n$, let $|\alpha| = \alpha_1 + \cdots + \alpha_n$ and $x^\alpha := x_1^{\alpha_1} \cdots x_n^{\alpha_n}$ for $x \in \mathbb{R}^n$. Let

$$D^\alpha := \frac{\partial^{\alpha_1 + \cdots + \alpha_n}}{\partial x_1^{\alpha_1} \cdots \partial x_n^{\alpha_n}} \equiv \frac{\partial^\alpha}{\partial x^\alpha}.$$

Let $P(D) := \sum_\alpha a_\alpha \frac{\partial^\alpha}{\partial x^\alpha} = \sum_\alpha a_\alpha D^\alpha$.

The *formal adjoint* of $P(D)$ is given by

$$P^*(D) = \sum (-1)^\alpha \overline{a_\alpha} D^\alpha.$$

It is called the formal adjoint since by integration by parts we have

$$\langle P(D)f, g\rangle = \langle f, P^*(D)g\rangle \quad \text{for } f, g \in C_0^\infty(\Omega).$$

Here the inner product is defined by $\langle f, g\rangle = \int_\Omega f\overline{g}\, dx$. The norms are with respect to this inner product.

1 Local Solvability—Proof 1

Let $P(D)$ denote a nonzero linear partial differential operator with constant coefficients, of order m.

Let us start with a simple case of Hörmander's inequality. Let $n = 1$ and $\Omega = (0, 1)$ and $P(D) = d/dx$. We wish to show that there exists $C > 0$ such that $\|\varphi'\| \geq C \|\varphi\|$ for all $\varphi \in C_0^\infty(0, 1)$.

The key trick is to observe the algebraic identity:

$$\langle (x\varphi)', \varphi \rangle = \langle x\varphi', \varphi \rangle + \langle \varphi, \varphi \rangle .$$

Further, it follows by integration by parts,

$$\langle (x\varphi)', \varphi \rangle = - \langle x\varphi, \varphi' \rangle .$$

Hence we have

$$\langle \varphi, \varphi \rangle = - \langle x\varphi', \varphi \rangle - \langle x\varphi, \varphi' \rangle .$$

We apply Cauchy-Schwarz inequality and use the fact $|x| < 1$ to get

$$\|\varphi\|^2 \leq 2 \|\varphi\| \|\varphi'\| ,$$

from which it follows that $\|\varphi'\| \geq \frac{1}{2} \|\varphi\|$.

Theorem 1 (Hörmander's Inequality)**.** *For every bounded open set Ω in $\mathbb{R}^n$, there exists a constant $C > 0$ such that for every $\varphi \in C_0^\infty(\Omega)$,we have*

$$\|P(D)\varphi\| \geq C \|\varphi\| . \tag{2}$$

We may take $C = |P|_m K_{m,\Omega}$ where

$$|P|_m = \max\{|a_\alpha|; |\alpha| = m\}$$

and $K_{m,\Omega}$ depends only on m and the diameter of Ω.

Proof. Given a differential operator $P(D)$ of order m, we define $P_j(D)$ by the formula

$$P(D)(x_j\varphi) = x_j P(D)\varphi + P_j(D)\varphi. \tag{3}$$

Note that the operator $P_j(D)$ is zero iff $P(D)$ does not involve any differentiation w.r.t. x_j. Order of $P_j(D)$ is strictly less than m, provided that it is non-zero.

Let $A := \sup_\Omega |x|$. By induction we shall show that

$$\|P_j(D)\varphi\| \leq 2mA \|P(D)\varphi\| . \tag{4}$$

Before proceeding to a proof of (4), we make two observations: (i) The definition of $P_j(D)$ along with (4) yields

$$\|P(D)(x_j\varphi)\| \leq (2m + 1)A \|P(D)\varphi\| .$$

(ii) The second one is a well-known property of normal operators:

$$\begin{aligned} \|P(D)\varphi\|^2 &= \langle P(D)\varphi, P(D)\varphi \rangle \\ &= \langle \varphi, P^*(D)P(D)\varphi \rangle \\ &= \langle \varphi, P(D)P^*(D)\varphi \rangle \\ &= \langle P^*(D)\varphi, P^*(D)\varphi \rangle \\ &= \|P^*(D)\varphi\|^2 . \end{aligned}$$

We now prove (4). It is trivial for $m = 0$. Let us assume that (4) holds true for all differential operators od order at most $m-1$. Let $P(D)$ be a differential operator of order m. We compute $\langle P(D)(x_j\varphi), P_j(D)\varphi\rangle$ in two different ways. From the defintion of $P_j(D)$ we have

$$\langle P(D)(x_j\varphi), P_j(D)\varphi\rangle = \langle x_j P(D)\varphi, P_j(D)\varphi\rangle + \langle P_j(D)\varphi, P_j(D)\varphi\rangle \,. \tag{5}$$

By integration by parts and the commutativity of $P^*(D)$ and $P(D)$, we obtain

$$\langle P(D)(x_j\varphi), P_j(D)\varphi\rangle = \left\langle P_j^*(D)(x_j\varphi), P^*(D)\varphi\right\rangle . \tag{6}$$

From (5) and (6) we find

$$\| P_j(D)\varphi\|^2 = \left\langle P_j^*(D)(x_j\varphi), P^*(D)\varphi\right\rangle - \langle x_j P(D)\varphi, P_j(D)\varphi\rangle \,. \tag{7}$$

By the above two observations and by the induction hypothesis, we get

$$\left\| P_j^*(D)(x_j\varphi)\right\| \leq (2m-1)A \,\| P_j(D)\varphi\| \,. \tag{8}$$

By Cauchy-Schwarz,

$$|\,\langle x_j P(D)\varphi, P_j(D)\varphi\rangle\,| \leq A \,\| P(D)\varphi\| \,\| P_j(D)\varphi\| \,. \tag{9}$$

Using (9) and (8) in (7), we get (4).

If $P(D)$ is of order $m \geq 1$, there exists j such that $P_j(D)$ is of order $m-1$. Observe that $|P_j|_{m-1} \geq |P|_m$. The theorem follows then immediately from induction. □

Corollary 2. *If Ω is a bounded open set in $\mathbb{R}^n$, then for any $g \in L^2(\Omega)$, there exists a weak solution $u \in L^2(\Omega)$ such that $P(D)u = g$.*

Proof. If $P(D)u = g$, then for all $\varphi \in C_0^\infty(\Omega)$, we have $\langle g, \varphi\rangle = \langle u, P^*(D)\varphi\rangle$.

Let $H_0 := \{\psi \in C_0^\infty(\Omega) : \psi = P^*(D)\varphi \text{ for some } \varphi \in C_0^\infty(\Omega)\}$. Hörmander's inequality implies that the map $\psi \mapsto \langle g, \varphi\rangle$ is well-defined (proof?), antilinear and continuous on E w.r.t. the L^2-norm. Hence it extends to a continuous linear map on the closure $\overline{E}$ of E in $L^2(\Omega)$. By Riesz representation theorem. there exists $u \in \overline{E} \subset L^2(\Omega)$ such that $\langle g, \varphi\rangle = \langle u, P^*(D)\varphi\rangle$ for all $\varphi \in C_0^\infty(\Omega)$. □

2 Local Solvability—Proof 2

For any $\alpha \in \mathbb{R}^n$, solving $P(D)u = f$ on $B(0, R)$ is equivalent to solving

$$P(D+\alpha)v = g, \text{ where } v = e^{-i\alpha\cdot x}u \text{ and } g = e^{-i\alpha\cdot x}f. \tag{10}$$

To solve (10) on $B(0, R)$, we can use a cut-off function to assume that g is supported in $B(0, 3R/2)$. This reduces the problem to one on $\mathbb{T}^n$. The following result then implies the solvability of (1) on $B(0, R)$.

Theorem 3. *For almost all $\alpha \in A := \{(\alpha_1, \ldots, \alpha_n) : 0 \leq \alpha_j \leq 1\}$, the map $P(D+\alpha)\colon \mathcal{D}'(\mathbb{T}_n) \to \mathcal{D}'(\mathbb{T}_n)$ and $P(D+\alpha\colon C^\infty(\mathbb{T}_n) \to C^\infty(\mathbb{T}_n)$ are isomorphisms.*

Recall that the distributions on $\mathbb{T}_n$ are characterized by the growth of their Fourier coefficients: For, $k \in \mathbb{Z}^n$, if (a_k) is the Fourier coefficient of a distribution u on $\mathbb{T}_n$, then there exist constants C and N such that

$$|a_k| \leq C|k|^N \quad \text{for } k \in \mathbb{Z}^n.$$

Similarly, smooth functions on $\mathbb{T}_n$ are characterized by the decay of their Fourier coefficients, namely, they vanish must faster than any polynomial in k.

Therefore it suffices to establish the following

Theorem 4. *Let $P(\xi)$ be a polynomial of degree m on $\mathbb{R}^n$. For almost all $\alpha \in A$, there exist constants C and N such that*

$$|P(k+\alpha)^{-1}| \leq C|k|^N \quad \textit{for all } k \in \mathbb{Z}^n. \tag{11}$$

Reference

J-P. Rosay, *A Very Elementary Proof of the Malgrange-Ehrenpreis Theorem*, AMM Vol. 98, pp. 518-523, 1991.