An Overview of the Theory of Linear PDE

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Abstract

This is a Very Biased Overview of PDE and Rough List of Recurrent Themes in PDE. It was written as answer to a senior mathematicant who just wanted a bird's eyeview of the subject. This was written almost 2 decades ago and so may be obsolete.

Natural Questions

Given a PDE, the natural questions are: Existence, Uniqueness and Stability or "continuous dependence on the given data".

Some Classical Results

- Cauchy-Kowalesky — Local existence — Analytic data
- Homgren — uniqueness
- H. Lewy's example;
- Local Solvability of constant coefficient PDO's.
- Hörmander's Fundamental results.

Well-posedness (More on it in a later item!)

Types of equations

- Elliptic
- Parabolic
- Hyperbolic.

This is not an exhaustive classification if the dimension is > 2. Examples.

Kinds of Solutions

- Classical
- Weak Solutions
- Distributions

Kinds of Spaces in which one looks for solutions

- Hölder Spaces
- L^p Spaces
- Distributions
- Sobolev Spaces

The kinds of results one needs about these spaces

- Approximations — Mollifiers
- Dense subspaces — Mollifiers
- Extensions
- Traces
- Compact subsets.

Types of Problems

- Initial value problems such as Cauchy problems
- Boundary Value Problems such as Dirichlet, Neuman problem, and Mixed problems

At a general glance a standard association:

- Laplace/Elliptic — Dirichlet
- Heat/Parabolic — Initial Value
- Wave/Hyperblic — Cauchy Problem along non-characteristics

Well-posedness; Hadamard's example

Typical questions and results

- In elliptic case, regularity results, maximum principles. In boundary value problems, **Index**, Atiyah-Singer Index Theorem.
- In hyperbolic case, propagation singularity, conservation of energy
- In parabolic case, conservation of energy; Study of Eigenvalues of the "associated" elliptic operator and their asymptotics. Meenakshisundaram and Patodi.

Some Very Frequently used Tools/Tricks/Principles

- Separation of Variables — Needs eigen space decomposition/Fourier series
- Divergence Theorems
- Variational Techniques
- Maximum Principles
- Energy Methods
- Fourier Transform Techniques
- A Priori Inequalities
- Continuity Method

Some Recent Developments

- Pseudo-differential operators
- Fourier Integral Operators
- Micro-local Analysis