First Order PDE — The Method of Characteristics

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Consider the quasi-linear equation

$$\sum_{j=1}^{n} a_j(x,u)\partial_j u = b(x,u). \tag{1}$$

If u is a function of x, the graph of u is the hypersurface in $\mathbb{R}^{n+1}$ given by the parameterization

$$(x_1,\ldots,x_n) \mapsto (x, u(x)).$$

The tangent space to the graph of u is spanned by $(e_j, \partial_j u)$ so that any normal to the hypersurface is a scalar multiple of $(\partial_1 u, \ldots, \partial_n u, -1)$. Hence the equation (1) just says that the vector field

$$A(x,y) := (a_1(x,y),\ldots,a_n(x,y), b(x,y))$$

is tangent to the graph of u. This suggests us that we can reconstruct the graph of u by flowing along the integral curves of A. These curves are called the *characteristic curves* of the PDE.

Given a hypersurface $S \subset \mathbb{R}^n$ and a function h on S, the initial value (or the Cauchy) problem is to find a solution u of (1) such that $u = h$ on S. In geometric language, this means that the graph of the solution u must contain the hypersurface of the graph of h on S. In general, there is a geometric constraint on the hypersurface S.

To understand this, let us consider the PDE $\partial_1 u = 0$ on $\mathbb{R}^2$ with the initial condition $u(x,0) = h(x)$, that is, $S = \{y = 0\}$ and $u = h$ on S. Then a necessary condition is that $\partial_1 h = 0$ or h is a constant on S.

The geometric condition is that for $x \in S$, the vector $A(x) := (a_1(x, h(x), \ldots, a_n(x, h(x))$ should not be tangent to S at x. Assume that S has a parameterization $\varphi\colon \mathbb{R}^{n-1} \to \mathbb{R}^n$ given by $(s_1,\ldots,s_{n-1}) \mapsto \varphi(s)$. Then the geometric condition can be phrased as the analytic condition

$$\det \begin{pmatrix} \frac{\partial\varphi_1}{\partial s_1} & \cdots & \frac{\partial\varphi_1}{\partial s_{n-1}} & a_1(\varphi(s), h(\varphi(s))) \\ \vdots & & \vdots & \vdots \\ \frac{\partial\varphi_n}{\partial s_1} & \cdots & \frac{\partial\varphi_n}{\partial s_{n-1}} & a_n(\varphi(s), h(\varphi(s))) \end{pmatrix} \neq 0. \tag{2}$$

Theorem 1. *Let S be a hypersurface and the functions a_j, b be real valued C^1-functions. Suppose that the vector*

$$A(x) := (a_1(x, h(x), \ldots, a_n(x, h(x))$$

is not tangent to S at x for any $x \in S$. Then there exists a unique solution u of the initial value problem defined on a neighbourhood of S.

Proof. Since the graph of u must be the union of integral curves of

$$A(x, y) = (a_1(x, y), \ldots, a_n(x, y), b(x, y))$$

passing through the points of $S^* := \{(x, h(x)) : x \in S\}$, the uniqueness follows.

Any hypersurface can be covered by open charts, that is, open sets which admit parametrization, say, $s \mapsto \varphi(s)$. If we solve the problem on each of these, by uniqueness, they agree on the intersection of their domains and hence we get a solution on all of S. So, we assume that S admits a single parametrization, $s \mapsto \varphi(s)$.

For each $s \in \mathbb{R}^{n-1}$, consider the initial value problem

$$\begin{aligned} \frac{\partial x_j}{\partial t}(s, t) &= a_j(x, y) \qquad x_j(s, 0) = \varphi_j(s), \qquad 1 \le j \le n \\ \frac{\partial y}{\partial t}(s, t) &= b(x, y) \qquad y(s, 0) = h(\varphi(s)). \end{aligned}$$

This is a system of ODE in t with parameter s. By the fundamental theorem of ODE, there is a unique solution (x, y) defined for small values of $|t|$. This solution is C^1 in the variables (x, y).

Since the hypersurface satisfies the non-characteristic condition (2), we can appeal to the inverse function theorem to conclude that the map $(s, t) \mapsto x(s, t)$ is a C^1-diffeomorphism. Hence we have a C^1-inverse of this map so that s, t are C^1-functions of x. We set $u(x) := y(s(x), t(x))$. By the initial value condition of the system of ODE, it is obvious that $u = h$ on S. We now show that u satisfies the PDE:

$$\begin{aligned} \sum a_j \partial_j u &= \sum_{j=1}^{n} a_j \left(\sum_{k=1}^{n-1} \frac{\partial u}{\partial s_k} \frac{\partial s_k}{\partial x_j} + \frac{\partial u}{\partial t} \frac{\partial t}{\partial x_k} \right) \\ &= \sum_{k=1}^{n-1} \frac{\partial u}{\partial s_k} \sum_{j=1}^{n} a_j \frac{\partial s_k}{\partial x_j} + \frac{\partial u}{\partial t} \sum_{j=1}^{n} a_j \frac{\partial t}{\partial x_j} \\ &= \sum_{k=1}^{n-1} \frac{\partial u}{\partial s_k} \sum_{j=1}^{n} \frac{\partial x_j}{\partial t} \frac{\partial s_k}{\partial x_j} + \frac{\partial u}{\partial t} \sum_{j=1}^{n} \frac{\partial x_j}{\partial t} \frac{\partial t}{\partial x_j} \\ &= \sum_{k=1}^{n-1} \frac{\partial u}{\partial s_k} \frac{\partial s_k}{\partial t} + \frac{\partial u}{\partial t} \frac{\partial t}{\partial t} \\ &= 0 + b. \end{aligned}$$

This completes the proof. □

Ex. 2. Solve the following initial value problems by the method of characteristics.

	Equation	**Initial Value**	**Solution**
(a)	$u_t + au_x = 0, a \in \mathbb{R}$	$u(x,0) = h(x)$	$u(x,t) = h(x - at)$
(b)	$xu_x + u_y = y$	$u(x,0) = x^2$	$u(x,y) = y^2/2 + x^2e^{-2y}$
(c)	$u_x + 2u_y = u^2$	$u(x,0) = h(x)$	$u(x,y) = \frac{h(x-y/2)}{1-(y/2)h(x-y/2)}$
(d)	$u_x + u_y = 1$	$u(x,0) = f(x)$	$u(x,y) = y + f(x - y)$
(e)	$xu_x + yu_y = u + 1$	$u(x,x^2) = x^2$	$u(x,y) = y + (x^2/y) - 1$
(f)	$xu_x + yu_y + u_z = 3u$	$u = \varphi(x,y)$ on $z = 0$	$u(x,y,z) = \varphi(xe^{-z}, ye^{-2z})e^{3z}$
(g)	$u_x + u_y = u$	$u = \cos x$ on $y = 0$	$u(x,y) = e^y \cos(x - y)$.