Fundamental Solutions for PDO with Constant Coefficients

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Introduction

We fix up a basic amount of notation to make our introduction intelligible.

We let $D_k := \frac{1}{i}\frac{\partial}{\partial x_k} = -i\frac{\partial}{\partial x_k}$. For $\alpha := (\alpha_1, \ldots, \alpha_n) \in \mathbb{Z}_+^n$, we let

$$D^\alpha := D_1^{\alpha_1} \cdots D_n^{\alpha_n} = \frac{(-i)^{|\alpha|}\partial^{|\alpha|}}{\partial x_1^{\alpha_1} \cdots \partial x_n^{\alpha_n}}$$

where $|\alpha| := \alpha_1 + \cdots + \alpha_n$.

A partial differential operator is an expression of the form $P(D) := \sum_{|\alpha|\leq m} a_\alpha D^\alpha$ where a_α are functions. It is said to be with constant coefficients if $a_\alpha \in \mathbb{C}$. Our main concern here will be with this class of operators.

A distribution E is said to be a *fundamental solution* of $P(D)$ if $P(D)E = \delta$, the Dirac distribution. A celebrated theorem of Malgrange-Ehrenpreis asserts the existence of "tempered' fundamental solutions for this class of operators. The interest in fundamental solutions lies in the fact that the knowledge of their existence allows us to solve the equations of the type $P(D)u = f$ where f is the given distribution with compact support. We need only take $u := E \star f$, the convolution of f with E! In particular, it asserts the local solvability of equation of the form $P(D)u = f$ in (distribution sense). They are also important for an entirely different reason. There was a time when mathematical community believed that any "reasonable" partial differential equation (say, with smooth coefficients) will have some kind of a solution. H. Levy sprang a surprise by producing a partial differential equation which admitted no solution, not even in the distribution sense! You can now fathom the relief of sigh heaved by the mathematical community when Malgrange-Ehrenpreis result appeared on the horizon!

If u is a distribution and φ is a test function, we shall use the standard pairing notation:

$$(u, \varphi) := u(\varphi)$$

This should not be confused with "inner product" or any such thing, if u happens to be a function!

The aims of these notes are (i) to exhibit fundamental solutions of some of the classical differential operators, (ii) to prove the local solvability of constant differential operators and

(iii) to outline a short proof of Malgrange-Ehrenpreis Theorem which asserts the existence of fundamental solutions for such operators. While they serve the purpose of giving very explicit and concrete fundamental solutions of operators of mathematical physics, such as Laplace, Heat and Wave , they also provide a good introduction to some of the ubiquitous tricks of hard-analysis in general, and PDE in particular.

In the penultimate section, we shall prove the local solvability of constant coefficient partial differential operators. More precisely, we shall prove that the equation $P(D)u = f$ in a bounded domain Ω of $\mathbb{R}^n$ has an L^2 solution. In the last section, we shall outline a very short proof of Malgrange-Ehrenpreis theorem.

1 Preliminaries — Integration-by-parts

Let Ω be a bounded domain in $\mathbb{R}^n$ with smooth boundary $S \equiv \partial\Omega$. Let $F \in C^1(\Omega, \mathbb{R}^n)$ and $F \in C(\overline{\Omega}, \mathbb{R}^n)$. Recall that the divergence of the vector field F is defined by

$$\operatorname{div} F \equiv \nabla \cdot F := \sum_j \partial_j F_j \qquad \text{where } F_j \text{ is the } j\text{-th component of } F.$$

We let $\nu = (\nu_1, \ldots, \nu_n)$ denote the unique outgoing unit normal on S. If S is locally given by $\varphi^{-1}0$ for a smooth function with $\nabla\varphi(p) \neq$ for $p \in S$, then we take $\nu(p) = \nabla\varphi(p)$. We let dS denote the surface measure (or area element) of S. For instance, if S is locally given as the graph of a function $(x_1, \ldots, x_{n-1}) \to \varphi(x_1, \ldots, x_{n-1})$, then

$$dS = (1 + \varphi_{x_1}^2 + \cdots + \varphi_{x_{n-1}}^2)^{1/2}\, dx_1 \cdots dx_{n-1}.$$

We recall the divergence theorem.

Theorem 1. *With the above notation we have*

$$\int_\Omega \operatorname{div} F dx = \int_S F \cdot \nu\, dS. \tag{1}$$

We deduce from this a lot of corollaries which will be useful later.

Theorem 2. *Let $f \in C^1(\Omega) \cap C(\overline{\Omega})$. Then*

$$\int_\Omega f_{x_i}\, dx = \int_S f\nu_i dS. \tag{2}$$

Proof. Take $F = (0, \ldots, 0, f, 0, \ldots, 0)$ with f at the i-th place in Eq. 1 □

Ex. 3. Let u be the distribution defined by the characteristic function χ_Ω of Ω. Compute $\frac{\partial u}{\partial x_j}$. *Answer:* This is precisely (2)!

Theorem 4 (Integration-by-parts)**.** *Let $f, g \in C^1(\Omega) \cap C(\overline{\Omega})$. Then*

$$\int_\Omega f_{x_i} g\, dx = -\int_\Omega f g_{x_i}\, dx + \int_S f g \nu_i\, dS. \tag{3}$$

If $fg = 0$ on S (in particular if one of them has compact support in Ω), we have

$$\int_\Omega f_{x_i} g\, dx = -\int_\Omega f g_{x_i}\, dx \tag{4}$$

Proof. Apply Eq. 2 to fg. □

Corollary 5 (Green's Identities). *Let $u, v \in C^2(\Omega) \cap C^1(\overline{\Omega})$. Then*

(i) *Gauss Law:*

$$\int_\Omega \Delta u = \int_S \frac{\partial u}{\partial \nu}\, dS. \tag{5}$$

(ii) *First Green's Identity:*

$$\int_\Omega \nabla u \cdot \nabla v \, dx = -\int_\Omega u \Delta v \, dx + \int_S \frac{\partial v}{\partial \nu} u \, dS. \tag{6}$$

(iii) *Second Green's Identity:*

$$\int_\Omega (u\Delta v - v\Delta u)\, dx = \int_S \left(u\frac{\partial v}{\partial \nu} - v\frac{\partial u}{\partial \nu} \right) dS. \tag{7}$$

Proof. Using Eq. 3 with u_{x_i} in place of u and $v = 1$ we see that

$$\int_\Omega u_{x_i x_i}\, dx = \int_S u_{x_i} \nu_i \, dS.$$

Summing over i yields (i).

To prove (ii), invoke Eq. 3 with $f = u$ and $g = v_{x_i}$.

Interchanging u and v in Eq. 6 and subtracting will result in (iii). □

We recall the polar coordinates on $\mathbb{R}^n$: For $x \in \mathbb{R}^n$, $x \neq 0$, we have the polar coordinates $x = r\xi$, where $r = \|x\|$ and $\xi := \|x\|^{-1} x \in S(0,1)$ in the unit sphere. With respect to this decomposition, we have the volume element

$$dx_1 \cdots dx_n = r^{n-1} dr dS(\xi)$$

where dS is the surface measure on the unit sphere. In particular, for any $f \in L^1(\mathbb{R}^n)$, we have

$$\int_{\mathbb{R}^n} f(x)\, dx = \int_0^\infty \int_{\|\xi\|=1} f(r\xi) r^{n-1}\, dr\, dS(\xi).$$

Notation: In the sequel, ω_n denotes the surface measure (or area) of the unit sphere on $\mathbb{R}^n$.

We use this to answer the following question: When is the function $x \mapsto |x|^\lambda$ locally integrable in a neighbourhood of 0 in $\mathbb{R}^n$? To answer this, we first of all observe that the function $x \mapsto |x|^\lambda$ is integrable in, say, $(0, R)$ iff

$$\int_0^R x^\lambda \, dx = \lim_{\varepsilon \to 0} \int_\varepsilon^R x^\lambda \, dx = [\frac{R^{\lambda+1}}{\lambda+1} - \frac{\varepsilon^{\lambda+1}}{\lambda+1}] \to \text{ a finite limit,}$$

that is, iff $\lambda > -1$. To answer the question in $\mathbb{R}^n$, we compute

$$\int_{|x|\leq R} |x|^\lambda dx = \int_0^R \int_{|\xi|=1} r^\lambda r^{n-1}\, dr\, dS(\xi) = \omega_n \int_0^R r^{\lambda+n-1}\, dr.$$

From the one variable discussion, we get the following:

Proposition 6. *The function r^λ is locally integrable near the origin in R^n iff $\lambda > -n$.*

The following exercise is often needed.

Ex. 7. If u is any continuous function in $\mathbb{R}^n$, we let

$$M_u(x,r) := \frac{1}{\omega_n}\int_{|y|=1} u(x+ry)\,dS(y)$$

denote the *mean value* of u at x over the sphere $\{|x-z| = r\}$. Then $M_u(x,r) \to u(x)$ as $r \to 0$. *Hint:* Note that $u(x) := \frac{1}{\omega_n}\int_{|u|=1} u(x)dS(y)$ so that

$$|M_u(x,r) - u(x)| = |\frac{1}{\omega_n}\int_{|y|=1} (u(x+ry) - u(x))\,|.$$

Use the continuity of u.

2 Fundamental Solution of Ordinary Differential Operator

Ex. 8. Let $f \in C^\infty(\mathbb{R})$ and let H be he Heaviside function

$$H(x) = \begin{cases} 1 & x > 0 \\ 0 & x \le 0. \end{cases}$$

Then $x \mapsto fH(x)$ is locally integrable and hence defines a distribution. Show that $(fH)' = f(0)\delta + Hf'$.

Ex. 9 (Fundamental Solution of Linear Ordinary D.O.)**.** Let

$$Lu := u^{(m)} + a_{m-1}(t)u^{(m-1)} + \cdots + a_1(t)u' + a_0(t)u$$

be an m-th order ordinary differential operator. Let Z be the solution of the Homogeneous DE $LZ = 0$ with the initial conditions

$$Z(0) = Z'(0) = \cdots = Z^{(m-2)}(0) = 0, \text{ and } Z^{(m-1)}(0) = 1.$$

Then $E(t) := Z(t)H(t)$ is a fundamental solution of L. In particular,

Lemma 10. *$H(t)e^{-at}$ and $H(t)\frac{\sin at}{a}$ are fundamental solutions of $\frac{d}{dt} + a$ and $\frac{d^2}{dt^2} + a^2$ respectively.* □

3 Fundamental Solution of the Cauchy-Riemann Operator

Let $\frac{\partial}{\partial \bar z} := \frac{1}{2}(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y})$ be the Cauchy-Riemann operator. This is called so for the following reason. If $f = u + iv\colon \mathbb{C} \to \mathbb{C}$ is a function, then $\frac{\partial}{\partial \bar z} f = 0$ iff u and v satisfy the Cauchy-Riemann equations.

Let Ω be an open set with smooth boundary given by the curve $\gamma\colon s \mapsto (x(s), y(s))$, parametrized by its arc-length. Let $f \in C^1(\overline{\Omega})$. Let $F = (f, if)$. Then $\operatorname{div} F = 2\frac{\partial f}{\partial \overline{z}}$. Also, the unit normal of the boundary $\gamma = \partial\Omega$ is given by $(-y', x')$. Applying the divergence theorem to F, we obtain

$$\int_\Omega \frac{\partial f}{\partial \overline{z}}\, dxdy = \frac{1}{2}\int_\gamma (-fy' + ifx')\, ds = \frac{i}{2}\int_\gamma (f\, dx + if\, dy) = \frac{i}{2}\int_\gamma f\, dz. \tag{8}$$

Let $u \in C^1(\overline{\Omega})$ and $z_0 \in \Omega$. For sufficiently small $\varepsilon > 0$, let $\Omega_\varepsilon := \Omega \setminus B[z_0, \varepsilon]$. Note that $\frac{\partial}{\partial \overline{z}}(\frac{1}{z-z_0}) = 0$ in Ω_ε. We wish to apply (8) to the function $f = u/(z - z_0)$ in Ω_ε. Before proceeding, let us observe that on the part of the boundary of Ω_ε given by $\{|z - z_0| = \varepsilon\}$ the unit normal going outward, i.e. the one that keeps the domain to its left, is given by $(y', -x')$. This explains the negative sign of the first term on the right side of (10) below.

$$\int_{\Omega_\varepsilon} \frac{\partial}{\partial \overline{z}}(\frac{u(z)}{z - z_0})\, dx\, dy = \int_{\Omega_\varepsilon} \frac{\partial u}{\partial \overline{z}}(\frac{1}{z - z_0})\, dx\, dy \tag{9}$$

$$= \frac{-i}{2}\int_{|z-z_0|=\varepsilon} \frac{u(z)}{z - z_0} dz + \frac{i}{2}\int_{\partial\Omega} \frac{u(z)}{z - z_0} dz. \tag{10}$$

We compute the integral over the circle. We have $z = z_0 + \varepsilon e^{i\theta}$ for z on $S(z_0, \varepsilon)$. Using this we get

$$\begin{aligned} -\frac{i}{2}\int_{|z-z_0|=\varepsilon} &\frac{u(z)}{z - z_0}\, dz \\ &= -\frac{i}{2}\int_0^{2\pi} u(z_0 + \varepsilon e^{i\theta}) i\, d\theta \\ &= \frac{1}{2}2\pi \left(\frac{1}{2\pi}\int_0^{2\pi} u(z_0 + \varepsilon e^{i\theta})\, d\theta\right) \\ &\to \pi u(z_0) \qquad \text{as } \varepsilon \to 0, \end{aligned} \tag{11}$$

by Ex. 7.

Equating the right sides of (9) and (10), letting $\varepsilon \to 0$ and using (11) we get

$$\int_\Omega \frac{\partial u}{\partial \overline{z}} \frac{1}{z - z_0}\, dxdy = \pi u(z_0) + \frac{i}{2}\int_{\partial\Omega} \frac{u(z)}{z - z_0}\, dz. \tag{12}$$

If we now assume that $u \in C_c^1(\Omega)$, the second term on the right side of (12) is zero and so, we get

$$u(z_0) = \frac{1}{\pi}\int_\Omega \frac{\partial u}{\partial \overline{z}} \frac{1}{z - z_0}\, dxdy. \tag{13}$$

In particular, it follows that $\frac{1}{\pi z}$ is a fundamental solution of the Cauchy-Riemann operator.

4 Fundamental Solution of Laplace Operator

Definition 11. The function

$$\Phi(x) := \begin{cases} \frac{1}{2\pi}\log(|x|) & n = 2 \\ \frac{1}{(2-n)\omega_n}|x|^{2-n} & n \geq 3 \end{cases}$$

defined for nonzero $x \in \mathbb{R}^n$ is known as the *fundamental solution* of the Laplace operator $\Delta := \sum_{j=1}^n \frac{\partial^2}{\partial x_j^2}$.

To get an idea how Φ was arrived at, we seek a radial solution of $\Delta u = 0$, that is, a solution u such that $u(x) = u(|x|)$ for $x \in \mathbb{R}^n$. Employing the standard notation $r := (\sum_j x_j^2)^{1/2}$, we arrive at the following after easy computations where u is a function of r alone:

$$\begin{aligned} \frac{\partial u}{\partial x_j} &= \frac{\partial u}{\partial r}\frac{\partial r}{\partial x_j} = \frac{x_j}{r} u_r \\ \frac{\partial^2 u}{\partial x_j^2} &= u_{rr}\frac{x_j^2}{r^2} + u_r \left(\frac{1}{r} - \frac{x_j^2}{r^3} \right). \end{aligned}$$

Summing over j in the last equation we arrive at

$$\Delta u = u_{rr} + \frac{n-1}{r} u_r \qquad \text{for a radial function } u.$$

Letting $v = u_r$ we obtain the ODE $v_r + \frac{n-1}{r} v = 0$. We solve for v and then for u to arrive at $u(r) = A \log r + B$ or $u(r) = A r^{2-n} + B$ according as $n = 2$ or $n > 2$.

Theorem 12. *For $v \in C_c^\infty(\mathbb{R}^n)$, we have*

$$\int_{\mathbb{R}^n} \Phi(x) \Delta(v)\, dx = v(0). \tag{14}$$

Proof. We deal with the case when $n \geq 3$. The case when $n = 2$ is very similarly dealt with.

Let $\Omega_\varepsilon := B(0, R) \setminus B[0, \varepsilon]$ where support of u is contained in $B(0, R)$. Using polar coordinates, one easily shows that Φ is integrable in any bounded neighbourhood of 0. (See Proposition 6.) Hence, it follows that

$$\int_{\Omega_\varepsilon} \Phi \Delta v\, dx \to \int_{\Omega} \Phi \Delta v\, dx \quad \text{as } \varepsilon \to 0.$$

Since Φ is harmonic in Ω_ε, by Eq. 7 we have

$$\int_{\Omega_\varepsilon} \Phi \Delta v\, dx = \int_{|x|=\varepsilon} \left(\Phi \frac{\partial v}{\partial \nu} - v \frac{\partial \Phi}{\partial \nu} \right) dS. \tag{15}$$

The outward unit normal on $|x| = \varepsilon$ is $\nu = -\frac{x}{\varepsilon}$.

We look at the first term on the right side of Eq. 15. On $|x| = \varepsilon$, we have $\Phi x = \Phi(\varepsilon)$ so that by Gauss Law (Eq. 5), we have

$$\int_{|x|=\varepsilon} \Phi \frac{\partial v}{\partial \nu}\, dS = -\Phi(\varepsilon) \int_{|x|<\varepsilon} \Delta v\, dx. \tag{16}$$

The right side in absolute value is at most

$$\Phi(\varepsilon) \omega_n \varepsilon^n \max_{\mathbb{R}^n} |\Delta v| = \text{Constant} \times \varepsilon^{2-n} \varepsilon^n$$

which goes to 0 as $\varepsilon \to 0$.

We now evaluate the second term on the right side of Eq. 15. On $|x| = \varepsilon$, it is easily checked by a trivial computation that

$$\frac{\partial \Phi}{\partial \nu}(|x|) = \nabla \Phi \cdot \nu = \nabla \Phi \cdot \frac{-x}{\varepsilon} = -\frac{\partial \Phi}{\partial r}(\varepsilon).$$

Using this, we obtain

$$\frac{\partial \Phi}{\partial \nu}(|x|) = -\frac{\partial \Phi}{\partial r}(\varepsilon) = (n-2)\frac{1}{(2-n)\omega_n}\varepsilon^{1-n}, \text{ if } n \geq 3.$$

Thus we compute

$$\int_{|x|=\varepsilon} v \frac{\partial \Phi}{\partial \nu}\, dS = -\frac{1}{\varepsilon^{n-1}\omega_n}\int_{|x|=\varepsilon} v\, dS = -M_v(0,\varepsilon) \to -v(0), \qquad \text{as } \varepsilon \to 0, \tag{17}$$

by Ex. 7. □

5 Fundamental Solution of the Heat Operator

Let

$$E(x,t) := \frac{H(t)}{(2a\sqrt{\pi t})^n}\exp(-\frac{|x|^2}{4a^2 t}).$$

We claim that E is a fundamental solution of the Heat operator $\frac{\partial}{\partial t} - a^2\Delta$.

We indicate a heuristic reasoning for this choice of E. Recall that the Fourier transform of a rapidly decreasing function f (or more generally for $f \in L^1(\mathbb{R}^n)$) is defined by $\mathcal{F}_x f(\xi) \equiv \hat{f}(\xi) := \int_{\mathbb{R}^n} f(x) e^{-ix\cdot\xi}\, dx$ and the inverse Fourier transform (under suitable assumptions on f) is given by

$$\mathcal{F}_\xi^{-1}(x) = \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n} f(\xi) e^{ix\cdot\xi}\, d\xi.$$

A special case worth mentioning is the Fourier transform of $f(x) = e^{-t|x|^2}$ for $t > 0$:

$$\int_{\mathbb{R}^n} e^{-t|x|^2} e^{-ix\cdot\xi}\, dx = (\frac{\pi}{t})^{n/2} e^{-|\xi|^2/4t}.$$

It is well-known that the Fourier transform is an isomorphism on the Schwarz space $\mathcal{S}$ of rapidly decreasing functions and hence the Fourier transform of a tempered distribution u is defined by

$$(\mathcal{F}u, \varphi) := (u, \mathcal{F}\varphi), \qquad \varphi \in \mathcal{S}.$$

However, if the distribution has compact support, then its Fourier transform is given by $\mathcal{F}u(x) = (2\pi)^{-n/2}(u, e^{ixy})$.

To find a candidate for the solution of $\frac{\partial E}{\partial t} - a^2\Delta E = \delta_{t=0}\delta_{x=0}$, we take the Fourier transform in the x-variable alone. Letting $\tilde{E}(\xi,t)$ stand for $\mathcal{F}_x E(\xi,t)$, we obtain an ODE

$$\frac{d}{dt}\tilde{E}(\xi,t) + a^2|\xi|^2\tilde{E} = \delta_{t=0}.$$

Using Lemma 10, we see that

$$\tilde{E}(\xi, t) = H(t)e^{-a^2|\xi|^2 t}.$$

By taking the inverse Fourier transform $\mathcal{F}_\xi^{-1}\tilde{E}(\xi, t)$, we get E as above.

The function E is locally integrable in $\mathbb{R}^{n+1}$ since $E = 0$ for $t \leq 0$ and for $t > 0$ we have

$$\int_{\mathbb{R}^n} E(x,t)\, dx = \frac{1}{(2a\sqrt{\pi t})^n} \int_{\mathbb{R}^n} \exp(-\frac{|x|^2}{4a^2 t})\, dx = \prod_{j=1}^{n} \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-\xi^2}\, d\xi = 1. \tag{18}$$

For $t > 0$ and $x \in \mathbb{R}^n$, $E(x,t)$ is smooth. We therefore compute

$$\frac{\partial E}{\partial t} = \left(\frac{|x|^2}{4a^2t^2} - \frac{n}{2t}\right) E \tag{19}$$

$$\frac{\partial E}{\partial x_j} = -\frac{x_j}{2a^2 t} E \tag{20}$$

$$\frac{\partial^2 E}{\partial x_j^2} = \left(\frac{x_j^2}{4a^2t^2} - \frac{1}{2a^2 t}\right) E. \tag{21}$$

It follows from (19) and (21 that $E(x,t)$ satisfies the heat equation for $t > 0$:

$$\frac{\partial}{\partial t} E - a^2 \Delta E = 0. \tag{22}$$

Let $\varphi \in C_c^\infty(\mathbb{R}^{n+1})$. We now compute:

$$\begin{aligned}
(\frac{\partial E}{\partial t} - a^2\Delta E, \varphi) &= -(E, \frac{\partial \varphi}{\partial t}) - (E, a^2\Delta\varphi) \\
&= -\int_{\mathbb{R}^n}\int_0^\infty E(x,t)\left(\frac{\partial\varphi}{\partial t} + a^2\Delta\varphi\right) dxdt \\
&= -\lim_{\varepsilon\to 0}\int_{\mathbb{R}^n}\int_\varepsilon^\infty E(x,t)\left(\frac{\partial\varphi}{\partial t} + a^2\Delta\varphi\right) dxdt \\
&= \lim_{\varepsilon\to 0}\left[\int_{\mathbb{R}^n} E(x,\varepsilon)\varphi(x,\varepsilon)\, dx \right.\\
&\quad \left. + \int_\varepsilon^\infty \int_{\mathbb{R}^n} \left(\frac{\partial E}{\partial t} - a^2\Delta E\right)\varphi\, dxdt\right] \\
&\quad \text{(by integration by parts)} \\
&= \lim_{\varepsilon\to 0}\int_{\mathbb{R}^n} E(x,\varepsilon)\varphi(x,\varepsilon)\, dx.
\end{aligned} \tag{23}$$

To obtain the last equality, we used (22). We change the variables in (23). Put $y = \frac{x}{2a\sqrt{\varepsilon}}$. Then the integral on the right side of (23) becomes

$$\int_{\mathbb{R}^n} \pi^{-n/2} \exp(-|y|^2)\varphi(2a\sqrt{\varepsilon}, \varepsilon)\, dy.$$

In this integral, the function $e^{-|y|^2}$ is in $L^1(\mathbb{R}^n)$ and φ is bounded on $\mathbb{R}^n$ so that we can apply the dominated convergence theorem to conclude that

$$\int_{\mathbb{R}^n} \pi^{-n/2}\exp(-|y|^2)\varphi(2a\sqrt{\varepsilon}, \varepsilon)\, dy \to \pi^{-n/2}\int_{\mathbb{R}^n} \exp(-|y|^2)\varphi(0,0)\, dy = \varphi(0,0).$$

6 Fundamental Solution of Wave Operator

6.1 One Dimensional Wave Equation

We show that $E(x,t) := \frac{1}{2a}H(at - |x|)$ is a fundamental solution of the one dimensional wave operator. The function E is locally integrable in $\mathbb{R}^2$ and vanishes outside the future cone $\{(x,t) : |x| \le at\}$. Let $\varphi \in C_c^\infty(\mathbb{R}^2)$. We compute

$$\begin{aligned}
(\Box E, \varphi) &= (E, \Box\varphi) = \int E(x,t)\Box\varphi(x,t)\, dxdt \\
&= \int_{-\infty}^{\infty}\int_{\frac{|x|}{a}}^{\infty} \frac{\partial^2\varphi}{\partial t^2}\, dtdx - \frac{2}{a}\int_0^\infty \int_{-at}^{at} \frac{\partial^2\varphi}{\partial x^2}\, dxdt \\
&= -\frac{1}{2a}\int_{-\infty}^{\infty} \frac{\partial\varphi}{\partial t}(\frac{|x|}{a})\, dx - \frac{a}{2}\int_0^\infty \int_{-at}^{at} \left[\frac{\partial\varphi}{\partial x}(at,t) - \frac{\partial\varphi}{\partial x}(-at,t)\right] dt \\
&= -\frac{1}{2a}\int_0^\infty \frac{\partial\varphi}{\partial t}(x, \frac{|x|}{a})\, dx - \frac{a}{2}\int_0^\infty \int_{-at}^{at} \frac{\partial\varphi}{\partial x}(at,t) \\
&\quad - \frac{1}{2a}\int_0^\infty \frac{\partial\varphi}{\partial t}(-x, \frac{|x|}{a})\, dx - \frac{a}{2}\int_0^\infty \int_{-at}^{at} \frac{\partial\varphi}{\partial x}(-at,t)\, dt \\
&= -\frac{1}{2}\int_0^\infty \left[\frac{\partial\varphi}{\partial t}(at,t) + a\frac{\partial\varphi}{\partial x}(at,t)\right] dt \\
&\quad - \frac{1}{2}\int_0^\infty \left[\frac{\partial\varphi}{\partial t}(-at,t) - a\frac{\partial\varphi}{\partial x}(-at,t)\right] dt \\
&= -\frac{1}{2}\int_0^\infty \frac{d\varphi}{dt}(at,t)\, dt - \frac{1}{2}\int_0^\infty \frac{d\varphi}{dt}(-at,t)\, dt \\
&= \frac{1}{2}(\varphi(0,0) + \varphi(0,0)) = (\delta, \varphi).
\end{aligned}$$

6.2 Three Dimensional Wave Equation via Partial Fourier Transform

Recall that the Fourier transform is an isomorphism on the Schwarz space $\mathcal{S}$ of rapidly decreasing functions and hence the Fourier transform of a tempered distribution u is defined by

$$(\mathcal{F}u, \varphi) := (u, \mathcal{F}\varphi), \qquad \varphi \in \mathcal{S}.$$

However, if the distribution has compact support, then its Fourier transform is given by $\mathcal{F}u(x) = (u, e^{ixy})$.

We use this fact to compute the Fourier transform of the Dirac measure δ_{S_R} of the sphere with centre at the origin and radius R in $\mathbb{R}^3$. In terms of polar coordinates, the inner product is given by $\langle x, y\rangle = x \cdot y = r\rho\cos\theta$ and the surface measure of S_R is given by

$$dS = R^2 \sin\theta\, d\theta\, d\varphi.$$

In the computations below, since y is fixed, we may assume that y is nonzero and the parametric representation of the sphere is got as the surface of revolution got by revolving a half-circle in a plane of which the $y/|y|$- is the x-axis. Under this assumption, we have

$$\mathcal{F}\delta_{S_R} = (\delta_{S_R}, e^{-ixy}) = R^2 \int_0^{2\pi} \int_0^{\pi} e^{-iR\rho\cos\theta} \sin\theta \, d\theta \, d\varphi = 4\pi R \frac{\sin R\rho}{\rho}. \tag{24}$$

We wish to find a fundamental solution of the wave operator using the Fourier transform techniques.

$$\Box E = \delta(x,t) \quad \text{where } \Box := \frac{\partial^2}{\partial t^2} - a^2\Delta.$$

Here Δ is the Laplace operator in $\mathbb{R}^n$.

We take the Fourier transform of the above equation in the x-variable alone. We write $\mathcal{F}_x$ for the Fourier transform in the x-variable. We let $\tilde{E}(\xi,t) := \mathcal{F}_x(E(x,t))$. We have

$$\frac{\partial^2 \tilde{E}(\xi,t)}{\partial t^2} + a^2|\xi|^2\tilde{E}(\xi,t) = 1(\xi)\cdot\delta(t), \tag{25}$$

in an obvious notation. Using Lemma 10, we see that

$$\tilde{E}(\xi,t) := H(t)\frac{\sin a|\xi|t}{a|\xi|}$$

is a solution of (25). It follows that

$$E(x,t) := \mathcal{F}_\xi^{-1}[\tilde{E}(\xi,t)] = H(t)\mathcal{F}_\xi^{-1}\left(\frac{\sin a|\xi|t}{a|\xi|}\right)$$

is a fundamental solution of the wave operator.

Assume $n=3$. Using (24, we compute the above inverse Fourier transform and obtain

$$E(x,t) = \frac{H(t)}{4\pi a^2 t}\delta_{S_R} \qquad \text{where } R = at$$

as a fundamental solution of the wave operator in $\mathbb{R}^3$. That is,

$$(E,\varphi) = \frac{1}{4\pi a^2}\int_0^\infty \frac{1}{t}\left(\int_{S_{at}} \varphi(x,t)dS(x)\right)dt, \qquad \varphi \in C_c^\infty(\mathbb{R}^4).$$

7 L^2-Local Solvability on Bounded Domains

For $f,g \in C^1(\overline{\Omega})$ with $fg = 0$ on $\partial\Omega$, (in particular, if one of them has compact support in Ω) we have

$$\int_\Omega f\frac{\partial g}{\partial x_k}\,dx = -\int_\Omega g\frac{\partial f}{\partial x_k}\,dx. \tag{26}$$

Ex. 13. We let $D_k := \frac{1}{i}\frac{\partial}{\partial x_k} = -i\frac{\partial}{\partial x_k}$. Then Eq. 4 becomes

$$\int_\Omega u\overline{D_k v}\,dx = \int_\Omega D_k u\overline{v}\,dx \quad \text{i.e. } \langle u, D_k v\rangle = \langle D_k u, v\rangle\,. \tag{27}$$

Ex. 14. For $\alpha := (\alpha_1, \ldots, \alpha_n) \in \mathbb{Z}_+^n$, we let

$$D^\alpha := D_1^{\alpha_1} \cdots D_n^{\alpha_n} = \frac{(-i)^{|\alpha|}\partial^{|\alpha|}}{\partial x_1^{\alpha_1} \cdots \partial x_n^{\alpha_n}}$$

where $|\alpha| := \alpha_1 + \cdots + \alpha_n$.

Ex. 15. Let $A := \sum_{|\alpha| \le m} a_\alpha(x) D^\alpha$ with $a_\alpha \in C^m(\overline{\Omega})$ and $\varphi \in C_c^m(\Omega)$. Let

$$A^*\varphi := \sum_{|\alpha| \le m} D^\alpha(\overline{a_\alpha(x)})\varphi.$$

Then we have

$$\int_\Omega Au\overline{\varphi}\, dx = \int_\Omega u\overline{A^*\varphi}\, dx \quad \text{i.e. } \langle Au, \varphi\rangle = \langle u, A^*\varphi\rangle . \tag{28}$$

That is, A^* is the formal adjoint of A w.r.t. the L^2-inner product.

Ex. 16. If $u \in C(\Omega)$ is such that $\int_\Omega u\overline{\varphi} = 0$ for all $\varphi \in C_c^\infty(\Omega)$, then $u = 0$.

Ex. 17. Let Ω be a bounded domain in $\mathbb{R}^n$. Let A be a differential operator as in Ex. 15. We want to solve $Au = f$, f a function defined on Ω. Assuming the existence of a solution, we derive a necessary condition, called an "a priori inequality".

Assume that $u \in C^m(\overline{\Omega})$ so that $Au \in C(\overline{\Omega})$. Then $Au = f$ and Ex. 15 implies that

$$\langle f, \varphi\rangle = \langle Au, \varphi\rangle = \langle u, A^*\varphi\rangle \quad \text{for } \varphi \in C_c^\infty(\Omega).$$

Using Schwarz inequality we arrive at the *a priori inequality* (also known as Hörmander's inequality: In order that $Au = f$ has a solution in Ω we must have

$$|\langle f, \varphi\rangle| \le \|u\| \, \|A^*\varphi\| = C \, \|A^*\varphi\| , \quad \varphi \in C_c^\infty(\Omega). \tag{29}$$

Ex. 18. If A has constant coefficients, the above a priori condition is also sufficient *provided* that we interpret the notion of a solution of the equation $Au = f$ in the *weak* sense. We say that $u \in L^2(\Omega)$ is a weak solution of $Au = f$ if $\langle u, A^*\varphi\rangle = \langle f, \varphi\rangle$ holds for all $\varphi \in C_c^\infty(\Omega)$.

Note that, in view of Ex. 16, in case $u \in C^m(\Omega)$, (m is the 'degree' of A), is a weak solution of $Au = f$, then it is, in fact, a classical solution.

The rest of the notes leads to a proof of sufficiency of Eq. 29 and that Eq. 29 holds for A with constant coefficients.

Ex. 19. Assume that Eq. 29 is true. Then $Au = f$ has a weak solution. For, let

$$V := \{A^*\varphi : \varphi \in C_c^\infty(\Omega)\}.$$

Define $F \colon V \to \mathbb{C}$ by setting $F(v) := \langle \varphi, f\rangle$, if $v = A^*\varphi$. Then F is well-defined. It is a continuous linear functional on V and hence on $\overline{V} \subset L^2(\Omega)$. Hence there exists $\tilde{F} \colon L^2(\Omega) \to \mathbb{C}$ with $\left\|\tilde{F}\right\| = \|F\|$ by Hahn-Banach theorem (which is trivial for Hilbert spaces). By Riesz representation theorem, we can find $u \in L^2(\Omega)$ such that $\tilde{F}(v) = \langle v, u\rangle$ for $v \in L^2(\Omega)$ with $\|u\| = \left\|\tilde{F}\right\| = \|F\|$. This u is a weak solution as required.

Thus it remains to prove the a-priori inequality for a differential operator A with constant coefficients: $A = \sum a_\alpha D^\alpha$, $a_\alpha \in \mathbb{C}$.

We look at a special case when $n = 1$ and $A = \frac{d}{dt}$. Let $\varphi \in C_c^\infty(a, b)$. We want to prove that

$$\|\varphi\| \leq C \|\varphi'\| . \tag{30}$$

Ex. 20. This yields a proof of Eq. 30. Write

$$\varphi(x) = \int_a^x \varphi'(t)\, dt$$

and apply Schwartz inequality on the RHS:

$$|\varphi(x)| \leq (x - a)^{1/2} \|\varphi'\|_{L^2(a,x)} \leq (b - a)^{1/2} \|\varphi'\|_{L^2(a,b)} .$$

It follows that

$$\int_a^b \|\varphi(x)\|^2\, dx \leq \int_a^b (b - a) \|\varphi'\|^2\, dx = (b - a)^2 \|\varphi'\|^2 .$$

Thus Eq. 30 is established with $C = (b - a)$.

Ex. 21. We employ Fourier series to offer a second proof of Eq. 30. Without loss of generality, assume that $\varphi \in C^1[0, \pi]$ with $\varphi(0) = 0$. We extend φ as an odd function ψ on $[-\pi, \pi]$. Then the Fourier (sine) series of ψ converges uniformly and absolutely to ψ so that we can differentiate it term by term to conclude that

$$\psi'(t) = \sum_{n=1}^\infty n\hat{\psi}(n) \cos nt.$$

Observe that

$$\|\psi\|^2 = \sum_{n=1}^\infty |\hat{\psi}(n)|^2 \quad \text{and} \quad \|\psi'\|^2 = 4 \sum_{n=1}^\infty n^2 |\hat{\psi}(n)|^2$$

This yields Eq. 30 with $C = 2$ for $a = 0$ and $b = \pi$.

We now return to the general case. Let $P(D)$ denote a nonzero linear partial differential operator with constant coefficients, of order m.

Let us start with a simple case of Hörmander's inequality. Let $n = 1$ and $\Omega = (0, 1)$ and $P(D) = d/dx$. We wish to show that there exists $C > 0$ such that $\|\varphi'\| \geq C \|\varphi\|$ for all $\varphi \in C_0^\infty(0, 1)$.

The key trick is to observe the algebraic identity:

$$\langle (x\varphi)', \varphi \rangle = \langle x\varphi', \varphi \rangle + \langle \varphi, \varphi \rangle .$$

Further, it follows by integration by parts,

$$\langle (x\varphi)', \varphi \rangle = - \langle x\varphi, \varphi' \rangle .$$

Hence we have

$$\langle \varphi, \varphi \rangle = -\left\langle x\varphi', \varphi \right\rangle - \left\langle x\varphi, \varphi' \right\rangle .$$

We apply Cauchy-Schwarz inequality and use the fact $|x| < 1$ to get

$$\| \varphi \|^2 \le 2 \| \varphi \| \, \| \varphi' \| ,$$

from which it follows that $\| \varphi' \| \ge \frac{1}{2} \| \varphi \|$.

Theorem 22 (Hörmander's Inequality)**.** *For every bounded open set Ω in $\mathbb{R}^n$, there exists a constant $C > 0$ such that for every $\varphi \in C_0^\infty(\Omega)$,we have*

$$\| P(D)\varphi \| \ge C \| \varphi \| . \tag{31}$$

We may take $C = |P|_m K_{m,\Omega}$ where

$$|P|_m = \max\{|a_\alpha|; |\alpha| = m\}$$

and $K_{m,\Omega}$ depends only on m and the diameter of Ω.

Proof. Given a differential operator $P(D)$ of order m, we define $P_j(D)$ by the formula

$$P(D)(x_j\varphi) = x_j P(D)\varphi + P_j(D)\varphi. \tag{32}$$

Note that the operator $P_j(D)$ is zero iff $P(D)$ does not involve any differentiation w.r.t. x_j. Order of $P_j(D)$ is strictly less than m, provided that it is non-zero.

Let $A := \sup_\Omega |x|$. By induction we shall show that

$$\| P_j(D)\varphi \| \le 2mA \| P(D)\varphi \| . \tag{33}$$

Before proceeding to a proof of (33), we make two observations: (i) The definition of $P_j(D)$ along with (33) yields

$$\| P(D)(x_j\varphi) \| \le (2m+1)A \| P(D)\varphi \| .$$

(ii) The second one is a well-known property of normal operators:

$$\begin{aligned}
\| P(D)\varphi \|^2 &= \langle P(D)\varphi, P(D)\varphi \rangle \\
&= \langle \varphi, P^*(D)P(D)\varphi \rangle \\
&= \langle \varphi, P(D)P^*(D)\varphi \rangle \\
&= \langle P^*(D)\varphi, P^*(D)\varphi \rangle \\
&= \| P^*(D)\varphi \|^2 .
\end{aligned}$$

We now prove (33). It is trivial for $m = 0$. Let us assume that (33) holds true for all differential operators of order at most $m-1$. Let $P(D)$ be a differential operator of order m. We compute $\langle P(D)(x_j\varphi), P_j(D)\varphi \rangle$ in two different ways. From the definition of $P_j(D)$ we have

$$\langle P(D)(x_j\varphi), P_j(D)\varphi \rangle = \langle x_j P(D)\varphi, P_j(D)\varphi \rangle + \langle P_j(D)\varphi, P_j(D)\varphi \rangle . \tag{34}$$

By integration by parts and the commutativity of $P^*(D)$ and $P(D)$, we obtain

$$\langle P(D)(x_j\varphi), P_j(D)\varphi \rangle = \left\langle P_j^*(D)(x_j\varphi), P^*(D)\varphi \right\rangle . \tag{35}$$

From (34) and (35) we find

$$\| P_j(D)\varphi \|^2 = \left\langle P_j^*(D)(x_j\varphi), P^*(D)\varphi \right\rangle - \langle x_j P(D)\varphi, P_j(D)\varphi \rangle . \tag{36}$$

By the above two observations and by the induction hypothesis, we get

$$\left\| P_j^*(D)(x_j\varphi) \right\| \le (2m-1)A \, \| P_j(D)\varphi \| . \tag{37}$$

By Cauchy-Schwarz,

$$| \langle x_j P(D)\varphi, P_j(D)\varphi \rangle | \le A \, \| P(D)\varphi \| \, \| P_j(D)\varphi \| . \tag{38}$$

Using (38) and (37) in (36), we get (33).

If $P(D)$ is of order $m \ge 1$, there exists j such that $P_j(D)$ is of order $m-1$. Observe that $|P_j|_{m-1} \ge |P|_m$. The theorem follows then immediately by induction. □

8 Malgrange-Ehrenpreis Theorem

Theorem 23. *Let $P(D)$ be a constant coefficient partial differential operator in $\mathbb{R}^n$ of degree m. Let $\eta \in \mathbb{R}^n$ be such that the top degree term $P_m(\eta) \neq 0$. Then the distribution E defined by*

$$E := \frac{1}{\overline{P_m(\eta)}} \int_{\mathbb{T}^1} \lambda^m e^{\lambda\eta x} \mathcal{F}^{-1}\left(\frac{\overline{P(i\xi + \lambda\eta)}}{P(i\xi + \lambda\eta)} \right) \frac{d\lambda}{2\pi i \lambda}$$

is a fundamental solution of $P(D)$. Furthermore, $E/\cosh(\eta x)$ is tempered.

Proof. Recall that the zero set of a nonzero polynomial function on $\mathbb{R}^n$ is of (Lebesgue) measure 0. This entails

$$\frac{\overline{P(i\xi + \lambda\eta)}}{P(i\xi + \lambda\eta)} \in L^\infty(\mathbb{R}^n_\xi) \text{ for any fixed } \lambda \in \mathbb{C}.$$

Lebesgue's dominated convergence theorem shows that the map

$$\mathbb{T}^1 \longrightarrow \mathcal{S}'(\mathbb{R}^n_\xi) : \lambda \mapsto \frac{\overline{P(i\xi + \lambda\eta)}}{P(i\xi + \lambda\eta)}$$

is continuous. Since a continuous function with values in $\mathcal{D}'(\mathbb{R}^n)$ can be integrated over any compact set, we infer that E is well-defined. It is easily verified that the distribution $E/\cosh(\eta x)$ is tempered.

We now compute $P(D)E$:

$$\begin{aligned} P(D)E &= \frac{1}{\overline{P_m(\eta)}} \int_{\mathbb{T}^1} \lambda^m P(D) \left[e^{\lambda\eta x} \mathcal{F}^{-1}\left(\frac{\overline{P(i\xi + \lambda\eta)}}{P(i\xi + \lambda\eta)} \right) \right] \frac{d\lambda}{2\pi i \lambda} \\ &= \frac{1}{\overline{P_m(\eta)}} \int_{\mathbb{T}^1} \lambda^m e^{\lambda\eta x} \left[P(D + \lambda\eta) \mathcal{F}^{-1}\left(\frac{\overline{P(i\xi + \lambda\eta)}}{P(i\xi + \lambda\eta)} \right) \right] \frac{d\lambda}{2\pi i \lambda} \\ &= \frac{1}{\overline{P_m(\eta)}} \int_{\mathbb{T}^1} \lambda^m e^{\lambda\eta x} \mathcal{F}^{-1}\left(\overline{P(i\xi + \lambda\eta)} \right) \frac{d\lambda}{2\pi i \lambda} \\ &= \frac{1}{\overline{P_m(\eta)}} \int_{\mathbb{T}^1} \lambda^m e^{\lambda\eta x} \overline{P(D + \lambda\eta)} \delta \frac{d\lambda}{2\pi i \lambda}. \end{aligned}$$

By Taylors' theorem,

$$\overline{P(D+\lambda\eta)}\delta = \overline{\lambda^m}\,\overline{P_m(\eta)}\delta + \sum_{k=0}^{m-1} \overline{\lambda^k} Q_k(D)\delta,$$

where Q_k are certain polynomials. By the residue theorem, the integrals over the terms with factors $\overline{\lambda^k}$, $0 \le k \le m-1$ vanish. The integral over the leading term yields δ. □

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