Local Solvability of Constant Coefficient Differential Operators

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1 Integration by Parts Formula

Recall the integration by parts formula.

Let Ω be a bounded domain in $\mathbb{R}^n$ with smooth boundary $S \equiv \partial\Omega$. Let $F \in C^1(\Omega, \mathbb{R}^n)$ and $F \in C(\overline{\Omega}, \mathbb{R}^n)$. Recall that the divergence of the vector field F is defined by

$$\operatorname{div} F \equiv \nabla \cdot F := \sum_j \partial_j F_j \qquad \text{where } F_j \text{ is the } j\text{-th component of } F.$$

We let $\nu = (\nu_1, \ldots, \nu_n)$ denote the unique outgoing unit normal on S. If S is locally given by $\varphi^{-1}0$ for a smooth function with $\nabla\varphi(p) \neq$ for $p \in S$, then we take $\nu(p) = \nabla\varphi(p)$. We let dS denote the surface measure (or area element) of S. For instance, if S is locally given as the graph of a function $(x_1, \ldots, x_{n-1}) \to \varphi(x_1, \ldots, x_{n-1})$, then

$$dS = (1 + \varphi_{x_1}^2 + \cdots + \varphi_{x_{n-1}}^2)^{1/2}\, dx_1 \cdots dx_{n-1}.$$

We recall the divergence theorem.

Theorem 1. *With the above notation we have*

$$\int_\Omega \operatorname{div} F dx = \int_S F \cdot \nu\, dS. \tag{1}$$

We deduce from this a couple of corollaries which will be useful later.

Theorem 2. *Let $f \in C^1(\Omega) \cap C(\overline{\Omega})$. Then*

$$\int_\Omega f_{x_i}\, dx = \int_S f \nu_i dS. \tag{2}$$

Proof. Take $F = (0, \ldots, 0, f, 0, \ldots, 0)$ with f at the i-th place in Eq. 1 □

Theorem 3 (Integration-by-parts)**.** *Let $f, g \in C^1(\Omega) \cap C(\overline{\Omega})$. Then*

$$\int_\Omega f_{x_i} g\, dx = -\int_\Omega f g_{x_i}\, dx + \int_S f g \nu_i\, dS. \tag{3}$$

If $fg = 0$ on S (in particular if one of them has compact support in Ω), we have

$$\int_\Omega f_{x_i} g \, dx = -\int_\Omega f g_{x_i} \, dx. \tag{4}$$

Proof. Apply Eq. 2 to fg. □

Ex. 4. We let $D_k := \frac{1}{i}\frac{\partial}{\partial x_k} = -i\frac{\partial}{\partial x_k}$. Then Eq. 4 becomes

$$\int_\Omega u\overline{D_k v} \, dx = \int_\Omega D_k u \overline{v} \, dx \quad \text{i.e. } \langle u, D_k v\rangle = \langle D_k u, v\rangle . \tag{5}$$

Ex. 5. For $\alpha := (\alpha_1, \ldots, \alpha_n) \in \mathbb{Z}^n_+$, we let

$$D^\alpha := D_1^{\alpha_1} \cdots D_n^{\alpha_n} = \frac{(-i)^{|\alpha|}\partial^{|\alpha|}}{\partial x_1^{\alpha_1} \cdots \partial x_n^{\alpha_n}}$$

where $|\alpha| := \alpha_1 + \cdots + \alpha_n$.

Ex. 6. Let $A := \sum_{|\alpha|\le m} a_\alpha(x) D^\alpha$ with $a_\alpha \in C^m(\overline{\Omega})$ and $\varphi \in C^m_c(\Omega)$. Let

$$A^*\varphi := \sum_{|\alpha|\le m} D^\alpha(\overline{a_\alpha(x)})\varphi.$$

Then we have

$$\int_\Omega Au\overline{\varphi} \, dx = \int_\Omega u\overline{A^*\varphi} \, dx \quad \text{i.e. } \langle Au, \varphi\rangle = \langle u, A^*\varphi\rangle . \tag{6}$$

That is, A^* is the formal adjoint of A w.r.t. the L^2-inner product.

Ex. 7. If $u \in C(\Omega)$ is such that $\int_\Omega u\overline{\varphi} = 0$ for all $\varphi \in C^\infty_c(\Omega)$, then $u = 0$.

2 L^2-Local Solvability on Bounded Domains

Ex. 8. Let Ω be a bounded domain in $\mathbb{R}^n$. Let A be a differential operator as in Ex. 6. We want to solve $Au = f$, f a function defined on Ω. Assuming the existence of a solution, we derive a necessary condition, called an “a priori inequality”.

Assume that $u \in C^m(\overline{\Omega})$ so that $Au \in C(\overline{\Omega})$. Then $Au = f$ and Ex. 6 implies that

$$\langle f, \varphi\rangle = \langle Au, \varphi\rangle = \langle u, A^*\varphi\rangle \quad \text{for } \varphi \in C^\infty_c(\Omega).$$

Using Schwarz inequality we arrive at the *a priori inequality* (also known as Hörmander’s inequality: In order that $Au = f$ has a solution in Ω we must have

$$|\langle f, \varphi\rangle| \le \|u\| \, \|A^*\varphi\| = C\,\|A^*\varphi\|, \quad \varphi \in C^\infty_c(\Omega). \tag{7}$$

Ex. 9. If A has constant coefficients, the above a priori condition is also sufficient *provided* that we interpret the notion of a solution of the equation $Au = f$ in the *weak* sense. We say that $u \in L^2(\Omega)$ is a weak solution of $Au = f$ if $\langle u, A^*\varphi\rangle = \langle f, \varphi\rangle$ holds for all $\varphi \in C^\infty_c(\Omega)$.

Note that, in view of Ex. 7, in case $u \in C^m(\Omega)$, (m is the ‘degree’ of A), is a weak solution of $Au = f$, then it is, in fact, a classical solution.

The rest of the notes leads to a proof of sufficiency of Eq. 7 and that Eq. 7 holds for A with constant coefficients.

Ex. 10. Assume that Eq. 7 is true. Then $Au = f$ has a weak solution. For, let

$$V := \{A^*\varphi : \varphi \in C_c^\infty(\Omega)\}.$$

Define $F\colon V \to \mathbb{C}$ by setting $F(v) := \langle \varphi, f \rangle$, if $v = A^*\varphi$. Then F is well-defined. It is a continuous linear functional on V and hence on $\overline{V} \subset L^2(\Omega)$. Hence there exists $\tilde{F}\colon L^2(\Omega) \to \mathbb{C}$ with $\left\| \tilde{F} \right\| = \| F \|$ by Hahn-Banach theorem (which is trivial for Hilbert spaces). By Riesz representation theorem, we can find $u \in L^2(\Omega)$ such that $\tilde{F}(v) = \langle v, u \rangle$ for $v \in L^2(\Omega)$ with $\| u \| = \left\| \tilde{F} \right\| = \| F \|$. This u is a weak solution as required.

Thus it remains to prove the a-priori inequality for a differential operator A with constant coefficients: $A = \sum a_\alpha D^\alpha$, $a_\alpha \in \mathbb{C}$.

We look at a special case when $n = 1$ and $A = \frac{d}{dt}$. Let $\varphi \in C_c^\infty(a, b)$. We want to prove that

$$\| \varphi \| \leq C \left\| \varphi' \right\|. \tag{8}$$

Ex. 11. This yields a proof of Eq. 8. Write

$$\varphi(x) = \int_a^x \varphi'(t)\, dt$$

and apply Schwartz inequality on the RHS:

$$|\varphi(x)| \leq (x - a)^{1/2} \left\| \varphi' \right\|_{L^2(a,x)} \leq (b - a)^{1/2} \left\| \varphi' \right\|_{L^2(a,b)}.$$

It follows that

$$\int_a^b \| \varphi(x) \|^2 \, dx \leq \int_a^b (b - a) \left\| \varphi' \right\|^2 \, dx = (b - a)^2 \left\| \varphi' \right\|^2.$$

Thus Eq. 8 is established with $C = (b - a)$.

Ex. 12. We employ Fourier series to offer a second proof of Eq. 8. Without loss of generality, assume that $\varphi \in C^1[0, \pi]$ with $\varphi(0) = 0$. We extend φ as an odd function ψ on $[-\pi, \pi]$. Then the Fourier (sine) series of ψ converges uniformly and absolutely to ψ so that we can differentiate it term by term to conclude that

$$\psi'(t) = \sum_{n=1}^\infty n\hat{\psi}(n) \cos nt.$$

Observe that

$$\| \psi \|^2 = \sum_{n=1}^\infty |\hat{\psi}(n)|^2 \quad \text{and} \quad \left\| \psi' \right\|^2 = 4 \sum_{n=1}^\infty n^2 |\hat{\psi}(n)|^2$$

This yields Eq. 8 with $C = 2$ for $a = 0$ and $b = \pi$.

We now return to the general case. Let $P(D)$ denote a nonzero linear partial differential operator with constant coefficients, of order m.

Let us start with a simple case of Hörmander's inequality. Let $n = 1$ and $\Omega = (0, 1)$ and $P(D) = d/dx$. We wish to show that there exists $C > 0$ such that $\| \varphi' \| \geq C \| \varphi \|$ for all $\varphi \in C_0^\infty(0, 1)$.

The key trick is to observe the algebraic identity:

$$\left\langle (x\varphi)', \varphi \right\rangle = \left\langle x\varphi', \varphi \right\rangle + \langle \varphi, \varphi \rangle .$$

Further, it follows by integration by parts,

$$\left\langle (x\varphi)', \varphi \right\rangle = - \left\langle x\varphi, \varphi' \right\rangle .$$

Hence we have

$$\langle \varphi, \varphi \rangle = - \left\langle x\varphi', \varphi \right\rangle - \left\langle x\varphi, \varphi' \right\rangle .$$

We apply Cauchy-Schwarz inequality and use the fact $|x| < 1$ to get

$$\| \varphi \|^2 \leq 2 \| \varphi \| \| \varphi' \| ,$$

from which it follows that $\| \varphi' \| \geq \frac{1}{2} \| \varphi \|$.

Theorem 13 (Hörmander's Inequality)**.** *For every bounded open set Ω in $\mathbb{R}^n$, there exists a constant $C > 0$ such that for every $\varphi \in C_0^\infty(\Omega)$,we have*

$$\| P(D)\varphi \| \geq C \| \varphi \| . \tag{9}$$

We may take $C = |P|_m K_{m,\Omega}$ where

$$|P|_m = \max\{|a_\alpha|; |\alpha| = m\}$$

and $K_{m,\Omega}$ depends only on m and the diameter of Ω.

Proof. Given a differential operator $P(D)$ of order m, we define $P_j(D)$ by the formula

$$P(D)(x_j\varphi) = x_j P(D)\varphi + P_j(D)\varphi. \tag{10}$$

Note that the operator $P_j(D)$ is zero iff $P(D)$ does not involve any differentiation w.r.t. x_j. Order of $P_j(D)$ is strictly less than m, provided that it is non-zero.

Let $A := \sup_\Omega |x|$. By induction we shall show that

$$\| P_j(D)\varphi \| \leq 2mA \| P(D)\varphi \| . \tag{11}$$

Before proceeding to a proof of (11), we make two observations: (i) The definition of $P_j(D)$ along with (11) yields

$$\| P(D)(x_j\varphi) \| \leq (2m + 1)A \| P(D)\varphi \| .$$

(ii) The second one is a well-known property of normal operators:

$$\begin{aligned} \| P(D)\varphi \|^2 &= \langle P(D)\varphi, P(D)\varphi \rangle \\ &= \langle \varphi, P^*(D)P(D)\varphi \rangle \\ &= \langle \varphi, P(D)P^*(D)\varphi \rangle \\ &= \langle P^*(D)\varphi, P^*(D)\varphi \rangle \\ &= \| P^*(D)\varphi \|^2 . \end{aligned}$$

We now prove (11). It is trivial for $m=0$. Let us assume that (11) holds true for all differential operators of order at most $m-1$. Let $P(D)$ be a differential operator of order m. We compute $\langle P(D)(x_j\varphi), P_j(D)\varphi\rangle$ in two different ways. From the definition of $P_j(D)$ we have

$$\langle P(D)(x_j\varphi), P_j(D)\varphi\rangle = \langle x_j P(D)\varphi, P_j(D)\varphi\rangle + \langle P_j(D)\varphi, P_j(D)\varphi\rangle . \tag{12}$$

By integration by parts and the commutativity of $P^*(D)$ and $P(D)$, we obtain

$$\langle P(D)(x_j\varphi), P_j(D)\varphi\rangle = \left\langle P_j^*(D)(x_j\varphi), P^*(D)\varphi\right\rangle . \tag{13}$$

From (12) and (13) we find

$$\|P_j(D)\varphi\|^2 = \left\langle P_j^*(D)(x_j\varphi), P^*(D)\varphi\right\rangle - \langle x_j P(D)\varphi, P_j(D)\varphi\rangle . \tag{14}$$

By the above two observations and by the induction hypothesis, we get

$$\left\|P_j^*(D)(x_j\varphi)\right\| \le (2m-1)A\,\|P_j(D)\varphi\| . \tag{15}$$

By Cauchy-Schwarz,

$$|\langle x_j P(D)\varphi, P_j(D)\varphi\rangle| \le A\,\|P(D)\varphi\|\,\|P_j(D)\varphi\| . \tag{16}$$

Using (16) and (15) in (14), we get (11).

If $P(D)$ is of order $m\ge 1$, there exists j such that $P_j(D)$ is of order $m-1$. Observe that $|P_j|_{m-1}\ge |P|_m$. The theorem follows then immediately by induction. □

3 Malgrange-Ehrenpreis Theorem

Theorem 14. *Let $P(D)$ be a constant coefficient partial differential operator in $\mathbb{R}^n$ of degree m. Let $\eta\in\mathbb{R}^n$ be such that the top degree term $P_m(\eta)\neq 0$. Then the distribution E defined by*

$$E := \frac{1}{\overline{P_m(\eta)}}\int_{\mathbb{T}^1} \lambda^m e^{\lambda\eta x}\mathcal{F}^{-1}\left(\frac{\overline{P(i\xi+\lambda\eta)}}{P(i\xi+\lambda\eta)}\right)\frac{\partial\lambda}{2\pi\lambda}$$

is a fundamental solution of $P(D)$. Furthermore, $E/\cosh(\eta x)$ is tempered.

Proof. Recall that the zero set of a nonzero polynomial function on $\mathbb{R}^n$ is of (Lebesgue) measure 0. This entails

$$\frac{\overline{P(i\xi+\lambda\eta)}}{P(i\xi+\lambda\eta)} \in L^\infty(\mathbb{R}^n_\xi) \text{ for any fixed } \lambda\in\mathbb{C}.$$

Lebesgue's dominated convergence theorem shows that the map

$$\mathbb{T}^1 \longrightarrow \mathcal{S}'(\mathbb{R}^n_\xi) : \lambda \mapsto \frac{\overline{P(i\xi+\lambda\eta)}}{P(i\xi+\lambda\eta)}$$

is continuous. Since a continuous function with values in $\mathcal{D}'(\mathbb{R}^n)$ can be integrated over any compact set, we infer that E is well-defined. It is easily verified that the distribution $E/\cosh(\eta x)$ is tempered.

We now compute $P(D)E$:

$$\begin{aligned} P(D)E &= \frac{1}{\overline{P_m(\eta)}} \int_{\mathbb{T}^1} \lambda^m P(D) \left[e^{\lambda\eta x} \mathcal{F}^{-1} \left(\frac{\overline{P(i\xi+\lambda\eta)}}{P(i\xi+\lambda\eta)} \right) \right] \frac{d\lambda}{2\pi\lambda} \\ &= \frac{1}{\overline{P_m(\eta)}} \int_{\mathbb{T}^1} \lambda^m e^{\lambda\eta x} \left[P(D+\lambda\eta) \mathcal{F}^{-1} \left(\frac{\overline{P(i\xi+\lambda\eta)}}{P(i\xi+\lambda\eta)} \right) \right] \frac{d\lambda}{2\pi\lambda} \\ &= \frac{1}{\overline{P_m(\eta)}} \int_{\mathbb{T}^1} \lambda^m e^{\lambda\eta x} \mathcal{F}^{-1} \left(\overline{P(i\xi+\lambda\eta)} \right) \frac{d\lambda}{2\pi\lambda} \\ &= \frac{1}{\overline{P_m(\eta)}} \int_{\mathbb{T}^1} \lambda^m e^{\lambda\eta x} \overline{P(D+\lambda\eta)} \delta \frac{d\lambda}{2\pi\lambda}. \end{aligned}$$

By Taylors' theorem,

$$\overline{P(D+\lambda\eta)}\delta = \overline{\lambda^m}\,\overline{P_m(\eta)}\delta + \sum_{k=0}^{m-1} \overline{\lambda^k} Q_k(D)\delta,$$

where Q_k are certain polynomials. By the residue theorem, the integrals over the terms with factors $\overline{\lambda^k}$, $0 \le k \le m-1$ vanish. The integral over the leading term yields δ. □