Problems in the Method of Separation of Variables

S. Kumaresan
School of Math. and Stat.
University of Hyderabad
Hyderabad 500046
kumaresa@gmail.com

1 Preliminaries on Fourier Series

Given a function $f \in L^1[-\pi, \pi]$, the Fourier series of f is the formal sum

$$\frac{a_0}{2} + \sum_{k=1}^{\infty} (a_k \cos kx + b_k \sin kx)$$

where

$$a_k := \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos kx \, dx, \quad b_k := \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin kx \, dx.$$

Theorem 1 (Dirichlet)**.** *Let $f \in L^1[-\pi, \pi]$ be bounded, continuous and monotone. Then the Fourier series of f converges to f pointwise.* □

Theorem 2. *Let $f \in L^1[-\pi, \pi]$ be C^1 and $f(-\pi) = f(\pi)$. The the Fourier series of f converges to f uniformly on $[-\pi, \pi]$.* □

If f is defined on $[-p, p]$, we can speak of the adapted Fourier series of f by considering the function $f_p(x) := f(\frac{px}{\pi})$ on $[-\pi, \pi]$. The Fourier series then is of the form

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} a_k \cos \frac{k\pi x}{p} + b_k \sin \frac{k\pi x}{p},$$

where the coefficients are given by

$$a_k := \frac{1}{p} \int_{-p}^{p} f(x) \cos \frac{k\pi x}{p} \, dx, \quad b_k = ?.$$

Extending this argument, the Fourier series of a function f defined on $[a, b]$ is of the form

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} a_k \cos \frac{2k\pi x}{b-a} + b_k \sin \frac{2k\pi x}{b-a},$$

where

$$a_k := \frac{2}{b-a} \int_{a}^{b} f(x) \cos \frac{2k\pi x}{b-a} \, dx, \quad b_k = ?.$$

The separation of variables method is to seek a solution $u(x, y)$ of a PDE in two variables in the form $u(x, y) = X(x)Y(y)$ where X and Y depend only on x and y respectively. This form is substituted in the given PDE and results in ODE for X and Y. We then use common sense. This approach is quite computational, and I advise the students to work out at least five of the following exercises to attain a certain level of facility. **Warning:** The given solutions may be wrong!

2 Wave Equation

Ex. 3. Use the separation of variables to solve the wave equation

$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2}$$

on the interval $[0, L]$ subject to the conditions

$$u(0, t) = 0 = u(L, t), u(x, 0) = f(x), u_t(x, 0) = g(x).$$

Simplify the result to show that $u(x, t) = f(x + ct) + g(x - ct)$. Solution:

$$u(x, t) = \sum_{n=1}^{\infty}\left(F_n \cos(\frac{cn\pi t}{L}) + G_n \sin(\frac{cn\pi t}{L})\right)\sin(\frac{n\pi x}{L}),$$

where

$$F_n := \frac{2}{L}\int_0^L f(x)\sin\frac{n\pi x}{L}\,dx$$

and

$$G_n := \frac{2}{n\pi c}\int_0^L g(x)\sin\frac{n\pi x}{L}\,dx.$$

Ex. 4. Use the method of separation of variables to obtain a solution of the wave equation

$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2}$$

on the interval $x \in [0, L]$ subject to the conditions that

$$\begin{aligned} u(0, t) &= 0, \qquad u(L, t) = 0 \\ u(x, 0) &= \sin(\pi x/L) + \sin(2\pi x/L) \\ u_t(x, 0) &= 0. \end{aligned}$$

Solution: $u(x, t) = \sum_{n=1}^{2} D_n \cos\frac{n\pi ct}{L}\sin\frac{n\pi x}{L}$.

Ex. 5. Find a separable solution of the wave equation

$$\frac{\partial^2 u}{\partial x^2} = a^2\frac{\partial^2 u}{\partial t^2}$$

on $[0, 1]$ which satisfies the conditions

$$\begin{aligned} u(0, t) = 0, &\qquad u(1, t) = 0 \\ u_t(x, 0) &= 0 \end{aligned}$$

and the initial displacement $u(x,0)$ consists of the string being pulled a small distance δ form the equilibrium position at a point one third of the way along, the two sections of the string being straight lines. Solution: $u(x,t) = \sum_{n=1}^{\infty} C_n \sin n\pi x \cos \frac{n\pi t}{a}$ where $c_n = \frac{9\delta \sin \frac{n\pi}{3}}{n^2\pi^2}$.

Ex. 6. Solve the wave equation using the separation of variables on the interval $[0, L]$ in which the end points are fixed at zero, and the initial displacement of the string is also zero. The motion is set in place by the string having an initial velocity given by

$$u_t(x,0) = \sin(n\pi x/L), \qquad n \in \mathbb{Z}.$$

Solution: $u = \frac{L}{n\pi c} \sin \frac{n\pi ct}{L} \sin \frac{n\pi x}{L}$.

Ex. 7. Use the method of separation of variables to solve the wave equation

$$\frac{\partial^2 u}{\partial x^2} = a^2 \frac{\partial^2 u}{\partial t^2}$$

on $[0, 2]$ satisfying the boundary conditions

$$\frac{\partial u}{\partial x}(0,t) = \frac{\partial u}{\partial x}(2,t)$$

for all t and the initial conditions

$$\frac{\partial u}{\partial t}(x,0) = 0, \quad 0 < x < 2$$

and

$$u(x,0) = \begin{cases} kx & 0 \le x \le 1 \\ k(2-x) & 1 \le x \le 2. \end{cases}$$

Solution: $\sum_{n=1}^{\infty} D_n \cos \frac{n\pi x}{L}$ where $D_n = \frac{8k}{n^2\pi^2} \cos \frac{n\pi}{2} - \frac{4k}{n^2\pi^2} \cos n\pi - \frac{4k}{n^2\pi^2}$.

Ex. 8. Use the separation of variables to solve the damped wave equation

$$c^2 \frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial t^2} + \mu \frac{\partial u}{\partial t}$$

on the interval $[0, L]$ subject to the conditions:

$$\begin{aligned} u(0,t) = 0, &\qquad u(L,t) = 0 \\ u(x,0) &= \sin(\pi x/L) \\ u_t(x,0) &= 0. \end{aligned}$$

Solution: $u(x,t) = \exp(-\frac{\mu t}{2}) \sin \frac{\pi x}{L} \cos \frac{1}{2} \sqrt{\frac{4\pi^2 c^2}{L^2} - \mu^2 t}$.

3 Heat Equation

Ex. 9. Solve the heat equation

$$\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2} \tag{1}$$

subject to

$$\frac{\partial u}{\partial x}(0,t) = 0 = \frac{\partial u}{\partial x}(L,t)$$

and

$$u(x,0) = f(x) = \beta x, \qquad 0 < x < L.$$

Solution:

$$u(x,t) = \frac{\beta L}{2} - \sum_{m=1}^{\infty} \frac{4L\beta}{(2m-1)^2\pi^2} \cos\frac{(2m-1)\pi x}{L} \exp\left(-\frac{(2m-1)^2\pi^2 c^2 ?}{t^2}\right).$$

Observe that $u(x,t) \to (\beta L)/2$ which is the mean value of f on $[0,L]$.

Ex. 10. Using the method of separation of variables, find the solution of the heat equation (1) on $[0,\pi]$ subject to the boundary conditions:

(i) $u = 0$ when $x = 0$ and $x = \pi$ and $u \to 0$ as $t \to \infty$;

(ii) $u_x = 0$ when $x = 0$ and $x = \pi$ and $u \to 0$ as $t \to \infty$. Solution: (i) $u = \sum_{n=1}^{\infty} A_p e^{-\frac{p^2 t}{a^2}} \sin nx$ and (ii) $u = \sum_{n=1}^{\infty} A_p e^{-\frac{p^2 t}{a^2}} \cos nx$.

Ex. 11. Use the separation of variables to solve the heat equation (1) in the region $0 < x < \pi$ and $t > 0$ satisfying the boundary conditions (i) $u = 0$ when $x = 0, \pi$, (ii) $u \to 0$ as $t \to \infty$ and (iii) $u = x + 1$ when $t = 0$ for $0 < x < \pi$. *Hint:* Use Ex. 10. Solution:

$$u\frac{2}{\pi}\frac{1}{n}[1-(1+\pi)\cos n\pi]e^{-\frac{n^2 t}{a^2}} \sin nx.$$

4 Potential Equation

Ex. 12. Solve the potential equation

$$u_{xx} + u_{yy} = 0, \qquad (x,y) \in [0,a] \times [0,b]$$

subject to the Dirichlet boundary conditions

$$u(0,y) = 0 = u(a,y), \quad u(x,0) = 0, u(x,b) = f(x).$$

Solution:

$$u(x,y) := \sum_{n=1}^{\infty} C_n \sin\frac{n\pi x}{a} \sinh\frac{n\pi y}{a}$$

where $C_n \sinh\frac{n\pi b}{a} = \frac{2}{a}\int_0^a f(x) \sin\frac{n\pi x}{a}\, dx$.