Classification of 2nd Order PDE

S. Kumaresan
School of Math. and Stat.
University of Hyderabad
Hyderabad 500046
kumaresa@gmail.com

In the following the given PDE is assumed to be

$$au_{xx} + bu_{xy} + cu_{yy} = G(x, y, u, u_x, u_y).$$

Ex. 1. Let $\xi = \xi(x, y)$ and $\eta = \eta(x, y)$ be a system of new coordinates. Show that the given equation transforms to

$$Au_{\xi\xi} + Bu_{\xi\eta} + Cu_{\eta\eta} + \text{lower order terms}$$

where

$$\begin{aligned} A &= a(\xi_x)^2 + b\xi_x\xi_y + c(\xi_y)^2 \\ B &= 2a\xi_x\eta_x + b(\xi_x\eta_y + \xi_y\eta_x) + 2c\xi_y\eta_y \\ C &= a(\eta_x)^2 + b\eta_x\eta_y + c(\eta_y)^2. \end{aligned}$$

Ex. 2. Let $\xi = \xi(x, y)$ and $\eta = \eta(x, y)$ be a smooth system of coordinates. Show that

$$B^2 - 4AC = (b^2 - 4ac)(\xi_x\eta_y - \xi_y\eta_x)^2.$$

Hence conclude that the given PDE and the transformed PDE are of the same type.

Ex. 3. Let γ be non-characteristic for the given equation. Let u, $\frac{\partial u}{\partial \nu}$ be known along γ. Show that the second partial derivatives of u are uniquely determined along γ.

Ex. 4. A curve $\gamma(s) := (x(s), y(s))$ is characteristic for the given PDE iff its principal symbol

$$\sigma(\xi) := a(x, y)\xi_1^2 + b(x, y)\xi_1\xi_2 + c(x, y)\xi_2^2$$

vanishes at its normal vector.

Ex. 5. Let the given PDE be hyperbolic. Assume that the new coordinates ξ and η are such that $\xi = \text{constant}$ and $\eta = \text{constant}$ are characteristic. Show that $A = 0 = C$ in the notation of Ex. 1.

Ex. 6. When do you say that the given PDE is hyperbolic, parabolic or elliptic? Give an example of each type. Give an example of a PDE which exhibits all these types. (Tricomi's equation $u_{yy} - yu_{xx} = 0$ is one such.)

Ex. 7. What are the canonical forms of the given PDE?

Ex. 8. Classify according to type:
(a) $u_{xx} + 2yu_{xy} + xu_{yy} - u_x + u = 0$.
(b) $2xyu_{xy} + xu_y + yu_x = 0$.

Ex. 9. Classify according to type and find the characteristics of
(a) $2u_{xx} - 4u_{xy} - 6u_{yy} + u_x = 0$.
(b) $4u_{xx} + 12u_{xy} + 9u_{yy} - 2u_x + u = 0$.
(c) $u_{xx} - x^2yu_{yy} = 0$, $(y > 0)$.
(d) $e^{2x}u_{xx} + 2e^{x+y}u_{xy} + e^{2y}u_{yy} = 0$.

Ex. 10. Find the characteristics of $u_{xx} + 2u_{xy} + \sin^2 xu_y = 0$.

Ex. 11. Find the characteristics of $(1 - x^2)u_{xx} - u_{yy} = 0$ in the hyperbolic case.

Ex. 12. Transform the equation into a canonical form:
(a) $2u_{xx} - 4u_{xy} - 6u_{yy} + u_x = 0$.
(b) $4u_{xx} + 12u_{xy} + 9u_{yy} - 2u_x + u = 0$.

Ex. 13. Find the type, characteristic curves and the canonical form of the following equations. Use the canonical form to find the general solution.
(a) $xu_{xx} + 2x^2u_{xy} = u_x - 1$. Answer: $u(x, y) = x + f(x^2 - y) + g(y)$.
(b) $3u_{xx} + 10u_{xy} + 3u_{yy} = 0$. Answer: $u(x, y) = f(y - 3x) + g(y - x/3)$.

Ex. 14. Reduce to canonical from:
(a) $u_{xx} + 5u_{xy} + 6u_{yy} = 0$.
(b) $x^2u_{xx} - 2xyu_{xy} + y^2u_{yy} = e^x$. Answer: $u_{\eta\eta} = \frac{2\xi}{\eta^2}u_\xi + \frac{1}{\eta^2}\exp(\xi/\eta)$, where $\xi = xy$ and $\eta = y$.

Ex. 15. Classify and reduce the equation

$$u_{xx} - 2\sin xu_{xy} - \cos^2 xu_{xy} - \cos xu_y = 0$$

to canonical form and hence solve it. Answer: $u(x, y) = f(y - x - \cos x) + g(y + x - \cos x)$.

Ex. 16. Reduce the Tricomi equation $u_{xx} + xu_{yy} = 0$ in its domain of hyperbolicity to its canonical form. Answer: $u_{\xi\eta} = \frac{1}{6(\xi-\eta)}(u_\xi - u_\eta)$.

Ex. 17. Reduce the equation $yu_{xx} = (x + y)u_{xy} + xu_{yy} = 0$ to its canonical form where it is hyperbolic and solve it there.

Ex. 18. Reduce the following equations to a canonical form and hence find general solutions.
(a) $yu_{xx}+(x+y)u_{xy}+xu_{yy} = 0$, in the hyperbolic case. Answer: $u(x, y) = \frac{1}{\xi}\int f(\eta)\, d\eta+g(\xi)$ where $\xi = y - x$ and $\eta = y^2 - x^2$.
(b) $u_{xx} + 2u_{xy} + u_{yy} = 0$.
(c) $y^2u_{xx} - 2xyu_{xy} + x^2u_{yy} = \frac{y^2}{x}u_x + \frac{x^2}{y}u_y$.
(d) $y^2u_{xx} - 4xyu_{xy} + 3x^2u_{yy} - \frac{y^2}{x}u_x + \frac{x^2}{y}u_y = 0$.