Higher Dimensional Poisson Equation

S. Kumaresan
School of Math. and Stat.
University of Hyderabad
Hyderabad 500046
kumaresa@gmail.com

1 Integration-by-parts

Let Ω be a bounded domain in $\mathbb{R}^n$ with smooth boundary $S \equiv \partial\Omega$. Let $F \in C^1(\Omega, \mathbb{R}^n)$ and $F \in C(\overline{\Omega}, \mathbb{R}^n)$. Recall that the divergence of the vector field F is defined by

$$\operatorname{div} F \equiv \nabla \cdot F := \sum_j \partial_j F_j \qquad \text{where } F_j \text{ is the } j\text{-th component of } F.$$

We let $\nu = (\nu_1, \ldots, \nu_n)$ denote the unique outgoing unit normal on S. If S is locally given by $\varphi^{-1}0$ for a smooth function with $\nabla\varphi(p) \neq$ for $p \in S$, then we take $\nu(p) = \nabla\varphi(p)$. We let dS denote the surface measure (or area element) of S. For instance, if S is locally given as the graph of a function $(x_1, \ldots, x_{n-1}) \to \varphi(x_1, \ldots, x_{n-1})$, then

$$dS = (1 + \varphi_{x_1}^2 + \cdots + \varphi_{x_{n-1}}^2)^{1/2}\, dx_1 \cdots dx_{n-1}.$$

We recall the divergence theorem.

Theorem 1. *With the above notation we have*

$$\int_\Omega \operatorname{div} F dx = \int_S F \cdot \nu \, dS. \tag{1}$$

We deduce from this a lot of corollaries which will be useful later.

Theorem 2. *Let $f \in C^1(\Omega) \cap C(\overline{\Omega})$. Then*

$$\int_\Omega f_{x_i}\, dx = \int_S f \nu_i dS. \tag{2}$$

Proof. Take $F = (0, \ldots, 0, f, 0, \ldots, 0)$ with f at the i-th place in Eq. 1 □

Theorem 3 (Integration-by-parts)**.** *Let $f, g \in C^1(\Omega) \cap C(\overline{\Omega})$. Then*

$$\int_\Omega f_{x_i} g\, dx = -\int_\Omega f g_{x_i}\, dx + \int_S f g \nu_i\, dS. \tag{3}$$

Proof. Apply Eq. 2 to fg. □

Corollary 4 (Green's Identities)**.** *Let* $u, v \in C^2(\Omega) \cap C^1(\overline{\Omega})$*. Then*

(i) *Gauss Law:*

$$\int_\Omega \Delta u = \int_S \frac{\partial u}{\partial \nu}\, dS. \tag{4}$$

(ii) *First Green's Identity:*

$$\int_\Omega \nabla u \cdot \nabla v \, dx = -\int_\Omega u \Delta v \, dx + \int_S \frac{\partial v}{\partial \nu} u \, dS. \tag{5}$$

(iii) *Second Green's Identity:*

$$\int_\Omega (u\Delta v - v\Delta u)\, dx = \int_S \left(u\frac{\partial v}{\partial \nu} - v\frac{\partial u}{\partial \nu}\right) dS. \tag{6}$$

Proof. Using Eq. 3 with u_{x_i} in place of u and $v = 1$ we see that

$$\int_\Omega u_{x_i x_i}\, dx = \int_S u_{x_i} \nu_i \, dS.$$

Summing over i yields (i).

Invoke Eq. 3 with v_{x_i} in place of v.

Interchanging u and v in Eq. 5 and subtracting will result in (iii). □

2 Poisson's Equation in $\mathbb{R}^n$

Let $f \in C_c(\mathbb{R}^n)$. Our aim in this section is to solve the so-called Poisson's equation.

$$-\Delta u = f. \tag{7}$$

First of all we look for radial solutions of $\Delta u = 0$. It is (already/easily) seen that if u is radial, i.e. $u(x) = u(|x|) = u(r)$ where $r := (\sum_j x_j^2)^{1/2}$, then

$$\Delta u = \frac{\partial^2 u}{\partial r^2} + \frac{n-1}{r}\frac{\partial u}{\partial r} = u_{rr} + \frac{n-1}{r} u_r.$$

If we let $v := u_r$, then the above equation reduces to an ODE

$$v' = \frac{1-n}{r} v$$

which readily admits solution: $v(r) = \frac{\text{Constant}}{r^{n-1}}$. Consequently, if $r \neq 0$, we have

$$u(r) = \begin{cases} b \log r + c & n = 2 \\ \frac{b}{r^{n-2}} + c & n \geq 3, \end{cases}$$

where b and c are constants. This discussion motivates the following

Definition 5. The function

$$\Phi(x) := \begin{cases} \frac{1}{2\pi} \log(|x|) & n = 2 \\ \frac{1}{(2-n)\omega_n} r^{2-n} & n \geq 3 \end{cases}$$

defined for nonzero $x \in \mathbb{R}^n$ is known as the *fundamental solution* of the Laplace equation $\Delta u = 0$. Here ω_n is the surface measure of the unit sphere in $\mathbb{R}^n$.

The choice of constants will become clear in Section 4.(See especially the proof of Theorem 11). It will also explain why Φ is called a fundamental solution of Δ.

Ex. 6. Show the following:

$$|\nabla \Phi(x)| \leq \frac{C}{\|x\|^{n-1}}, \qquad (x \neq 0) \tag{8}$$

and

$$|D^2 \Phi(x)| \leq \frac{C}{\|x\|^{n}}, \qquad (x \neq 0) \tag{9}$$

for some $C > 0$. Here D^2 stands for the Hessian matrix of second order derivatives.

Remark 7. The function $\Phi(x)$ is harmonic for $x \neq 0$ so that if we shift the origin to y, the function $x \mapsto \Phi(x-y)$ is harmonic for $x \neq y$. If f is any function then $x \mapsto \Phi(x-y)f(y)$ is harmonic for any y and so is any finite linear combination for a finite set of y's.

This suggests the "convolution"

$$u(x) = \int_{\mathbb{R}^n} \Phi(x-y) f(y)\, dy, \tag{10}$$

may solve the Laplace equation $\Delta u = 0$.

Because of the estimate Eq. 9, we cannot justify the differentiation under the integral sign in the equation below:

$$\Delta u(x) = \int_{\mathbb{R}^n} \Delta_x \Phi(x-y) f(y)\, dy = 0.$$

In fact, this is wrong! See the next theorem.

Theorem 8 (Solution of Poisson's equation)**.** *Let $f \in C_c(\mathbb{R}^n)$ and let u be defined as in Eq. 10. Then $u \in C^2(\mathbb{R}^n)$ and $\Delta u = f$.*

Proof. Note that, by a change of variables $y \mapsto x - y$, we have

$$u(x) = \int_{\mathbb{R}^n} \Phi(x-y) f(y)\, dy = \int_{\mathbb{R}^n} \Phi(y) f(x-y)\, dy.$$

We use the second representation to compute the partial derivatives of u:

$$\frac{u(x+te_i) - u(x)}{t} = \int_{\mathbb{R}^n} \Phi(y) \left[\frac{f(x-y+te_i) - f(x-y)}{t} \right] dy.$$

Since f is C^2 and compactly supported, the partial derivatives $\frac{\partial f}{\partial x_i}$ are uniformly continuous on $\mathbb{R}^n$ so that

$$\frac{f(x-y+te_i)-f(x-y)}{t} \to \frac{\partial f}{\partial x_i}(y-x) \quad \text{uniformly on } \mathbb{R}^n$$

as $t \to 0$. Thus it follows that

$$\frac{\partial u}{\partial x_i} = \int_{\mathbb{R}^n} \Phi(y) \frac{\partial f}{\partial x_i}(x-y)\, dy \quad (1 \le i \le n).$$

One similarly shows that

$$\frac{\partial^2 u}{\partial x_i \partial x_j} = \int_{\mathbb{R}^n} \Phi(y) \frac{\partial^2 f}{\partial x_i \partial x_j}(x-y)\, dy \quad (1 \le i, j \le n).$$

Thus u is C^2. Also, since f is compactly supported, it follows that u, ∇u and $\nabla^2 u$ are bounded.

To compute Δu we use the most useful trick of integration theory. The trick is to split the integral into parts: one part where the integrand is well-behaved and the other where the integrand has bad behaviour. If one is lucky, the set where the integrand is badly behaved will have small measure (volume). See the estimate of the first term I_ε below. In our context the problem is the behaviour of Φ near the origin. Hence we proceed as follows. Let $\varepsilon > 0$ be given. Then

$$\Delta u(x) = \int_{B(0,\varepsilon)} \Phi(y) \Delta_x f(x-y)\, dy + \int_{\mathbb{R}^n \setminus B(0,\varepsilon)} \Phi(y) \Delta_x f(x-y)\, dy = I_\varepsilon + J_\varepsilon. \tag{11}$$

We estimate I_ε as follows:

$$|I_\varepsilon| \le C \left\| D^2 f \right\|_\infty \int_{B(0,\varepsilon)} |\Phi(y)|\, dy \le \begin{cases} C\varepsilon^2 |\log \varepsilon| & n = 2 \\ C\varepsilon^2 & n \ge 3. \end{cases} \tag{12}$$

To estimate J_ε we apply the integration by parts (following the dictum "If you have nothing else to do then integrate by parts!").

$$\begin{aligned} J_\varepsilon &= \int_{\mathbb{R}^n \setminus B(0,\varepsilon)} \Phi(y) \Delta_y f(x-y)\, dy \\ &= -\int_{\mathbb{R}^n \setminus B(0,\varepsilon)} \nabla \Phi(y) \nabla_y f(x-y)\, dy + \int_{\partial B(0,\varepsilon)} \Phi(y) \frac{\partial f}{\partial \nu}(x-y) dS(y) \\ &= J_{1\varepsilon} + J_{2\varepsilon}. \end{aligned} \tag{13}$$

A point to be noted here is that ν is the unit normal pointing outward to the domain $\mathbb{R}^n \setminus B(0,\varepsilon)$ so that it is the unit normal along $S(0,\varepsilon)$ pointing inward $B(0,\varepsilon)$. That is, $\nu(y) = -\frac{y}{\varepsilon}$ along $S(0,\varepsilon)$.

The second term of Eq. 13 is easily estimated:

$$|J_{2\varepsilon}| \le \| \nabla \Phi \|_\infty \int_{S(0,\varepsilon)} |\Phi(y)| dS(y) \le \begin{cases} C\varepsilon |\log \varepsilon| & n = 2 \\ C\varepsilon & n \ge 3. \end{cases} \tag{14}$$

We again resort to integration by parts to deal with the first term $J_{1\varepsilon}$ of Eq. 13.

$$\begin{aligned} J_{1\varepsilon} &= \int_{\mathbb{R}^n\setminus B(0,\varepsilon)} \Delta\Phi(y) f(x-y)\,dy - \int_{S(0,\varepsilon)} \frac{\partial\Phi}{\partial\nu}(y) f(x-y)\,dS(y) \\ &= -\int_{S(0,\varepsilon)} \frac{\partial\Phi}{\partial\nu}(y) f(x-y)\,dS(y). \end{aligned}$$

We compute $\nabla\Phi(y) = \frac{y}{\omega_n |y|^n}$ for $y \neq 0$. We already observed that $\nu(y) = -\frac{y}{\varepsilon}$ along $S(0,\varepsilon)$. As a result, we find that

$$\frac{\partial\Phi}{\partial\nu}(y) = \nu \cdot \nabla\Phi(y) = -\frac{1}{\omega_n \varepsilon^{n-1}}$$

on $S(0,\varepsilon)$. Thus,

$$J_{1\varepsilon} = \frac{1}{\omega_n \varepsilon^{n-1}} \int_{S(0,\varepsilon)} f(x-y)\,dS(y) = M_f(x,\varepsilon) \to f(x) \tag{15}$$

as $\varepsilon \to 0$. From Eq. 11—Eq. 15, we get what we wanted. □

3 An Integral Representation

We have already seen that solutions of the Laplace equation $\Delta u = 0$ which are of the form $u(x) = u(|x-\xi|)$ which depend on the distance from the point ξ are given by

$$u(x) = \begin{cases} \log(r), & n = 2 \\ \frac{1}{(2-n)\omega_n r^{n-2}}, & n \geq 3 \end{cases} \text{where } r = |x-\xi|. \tag{16}$$

Each of these functions have a singularity at ξ the nature of the singularity depending on the dimension n. We introduce the Newtonian potential

$$K(x,\xi) = \begin{cases} \frac{1}{2\pi}\log(r) & n = 2 \\ \frac{1}{(2-n)\omega_n r^{n-2}} & n \geq 3 \end{cases} \qquad r = |x-\xi|. \tag{17}$$

The aim of this section is to obtain an integral representation of a solution of the boundary value problem for the Poisson's equation

$$\Delta u(x) = f(x) \text{ on } \Omega \quad \& \quad u(x) = g(x) \quad \text{ on } \partial\Omega. \tag{18}$$

The starting point here is the second Green's identity Eq. 6. We shall apply it to a smooth function u and choose $v = K(x,\xi)$ for $\xi \in \Omega$. Because of the singularity at ξ, we cannot apply the identity directly to the domain Ω. We scoop out a small closed ball around ξ from Ω. Let $\Omega_\varepsilon := \Omega \setminus B[\xi,\varepsilon]$. In Ω_ε, $K(x,\xi)$ is harmonic in x so that $\Delta K(\cdot,\xi) = 0$ in Ω_ε. The boundary of Ω_ε is $\partial\Omega \cup S(\xi,\varepsilon)$. The outward unit normal on $S(\xi,\varepsilon)$ is given by $-\frac{x-\xi}{\varepsilon}$. Green's second identity now yields

$$\begin{aligned} -\int_{\Omega_\varepsilon} \Delta u(x) K(x,\xi)\,dx &= \int_S \left(u(x)\frac{\partial K(x,\xi)}{\partial\nu} - K(x,\xi)\frac{\partial u(x)}{\partial\nu} \right) dS(x) \\ &\quad - \int_{S(\xi,\varepsilon)} \left(u(x)(-\frac{1}{\omega_n \varepsilon^{n-1}} - \frac{1}{(n-2)\omega_n \varepsilon^{n-2}} \frac{\partial u(x)}{\partial\nu} \right) dS(x). \end{aligned} \tag{19}$$

We consider the behaviour of each term as $\varepsilon \to 0$.

The first integral (over $S(\xi,\varepsilon)$) of the second term is the spherical means of u over the sphere $S(\xi,\varepsilon)$ which by the continuity of u goes to $u(\xi)$ as $\varepsilon \to 0$.

We now deal with the second integral over $S(\xi,\varepsilon)$. Using Gauss Law Eq. 4, we estimate

$$|\int_{S(\xi,\varepsilon)} \frac{\partial u(x)}{\partial \nu} dS(x)| = |\int_{B(\xi,\varepsilon)} \Delta u(x)\, dx| \le (\frac{\omega_n \varepsilon^n}{n}) \max_{B(\xi,\varepsilon)} |\Delta u(x)| \tag{20}$$

From this it follows that

$$\lim_{\varepsilon\to 0} \frac{1}{\varepsilon^{n-2}} \int_{S(\xi,\varepsilon)} \frac{\partial u(x)}{\partial \nu}\, dS(x) = 0. \tag{21}$$

We have thus shown that the right side of Eq. 19 converges as $\varepsilon \to 0$ and therefore the left side dose converge. We have obtained the integral representation formula, Eq. 22 of the following theorem.

Theorem 9. *Let $u \in C^2(\Omega) \cap C^1(\overline{\Omega})$ and $\xi \in \Omega$. The we have*

$$u(\xi) = \int_{\partial\Omega} \left(K(x,\xi)\frac{\partial u(x)}{\partial \nu} - u(x)\frac{\partial K(x,\xi)}{\partial \nu} \right) dS(x) - \int_{\Omega} \Delta u(x) K(x,\xi)\, dx. \tag{22}$$

□

Remark 10. This representation expresses the value of a function u as in the theorem at a point $\xi \in \Omega$ *in terms of* Δu in Ω **and** both of u and $\frac{\partial u}{\partial \nu}$ on the boundary $\partial\Omega$. The required knowledge of the normal derivative of u on the boundary is the major defect of this formula. However, this formula is the stepping stone to the final solution of the Dirichlet problem for the Poisson equation.

4 Why Fundamental Solution?

Let δ denote the Dirac measure on $\mathbb{R}^n$: $\delta(E) = \begin{cases} 1 & 0 \in E \\ 0 & 0 \notin E \end{cases}$. We say that a locally integrable function E is a *fundamental solution* of a constant coefficient differential operator $P(D)$ if $P(D)E = \delta$ in the *distribution sense*: $\int_{\mathbb{R}^n} EP(-D)\varphi = \varphi(0)$ for all $\varphi \in C_c^\infty(\Omega)$.

Theorem 11. *For $v \in C_c^\infty(\mathbb{R}^n)$, we have*

$$\int_{\mathbb{R}^n} \Phi(x)\Delta(v)\, dx = v(0). \tag{23}$$

Proof. Let $\Omega_\varepsilon := B(0,R) \setminus B[0,\varepsilon]$ where support of u is contained in $B(0,T)$. Using polar coordinates, one easily shows that Φ is integrable in any bounded neighbourhood of 0. Hence, it follows that

$$\int_{\Omega_\varepsilon} \Phi\Delta v\, dx \to \int_{\Omega} \Phi\Delta v\, dx \quad \text{as } \varepsilon \to 0.$$

Since Φ is harmonic in Ω_ε, by Eq. 6 we have

$$\int_{|??|ve} \Phi \Delta v \, dx = \int_{|x|=\varepsilon} \left(\Phi \frac{\partial v}{\partial \nu} - v \frac{\partial \Phi}{\partial \nu} \right) dS. \tag{24}$$

The outward unit normal on $|x| = \varepsilon$ is $\nu = -\frac{x}{\varepsilon}$.

We look at the first term on the right side of Eq. 24. On $|x| = \varepsilon$, we have $\Phi x = \Phi(\varepsilon)$ so that by Gauss Law (Eq. 4), we have

$$\int_{|x|=\varepsilon} \Phi \frac{\partial v}{\partial \nu} \, dS = -\Phi(\varepsilon) \int_{|x|<\varepsilon} \Delta v \, dx. \tag{25}$$

The right side is absolute value is at most $\Phi(\varepsilon)\omega_n \varepsilon^n \max_{|x|\leq\varepsilon} |\Delta v|$ which goes to 0 as $\varepsilon \to 0$.

We now evaluate the second term on the right side of Eq. 24. On $|x| = \varepsilon$, we have

$$\frac{\partial \Phi}{\partial \nu}(|x|) = -\frac{\partial \Phi}{\partial r}(\varepsilon) = (n-2)C\varepsilon^{1-n}, \text{ if } n \geq 3.$$

Thus we compute

$$\int_{|x|=\varepsilon} v \frac{\partial \Phi}{\partial \nu} \, dS = C\varepsilon^{1-n} \int_{|x|=1} v \, dS = C\omega_n M_v(0, \varepsilon) \to v(0), \tag{26}$$

by the continuity of v. □