One Dimensional Wave Equation

S. Kumaresan
School of Math. and Stat.
University of Hyderabad
Hyderabad 500046
kumaresa@gmail.com

Ex. 1. Find the characteristics of the wave equation $u_{tt} - c^2 u_{xx} = 0$, transform it to a canonical form and find its general solution.

Ex. 2. Solve the initial value problem for the wave equation $u_{tt} - c^2 u_{xx} = 0$ with the Cauchy data $u(x, 0) = g(x)$ and $u_t(x, 0) = h(x)$. Answer is given by d'Alembert's formula:

$$u(x, t) = \frac{1}{2}[g(x + ct) - g(x - ct)] + \frac{1}{2c} \int_{x-ct}^{x+ct} h(s)\, ds.$$

Ex. 3. Explain the concepts 'domain of dependence', 'range of influence' and 'finite propagation speed' associated with the solution of the initial value problem of the wave equation with Cauchy data.

Ex. 4. Use the method of separation of variables to solve the initial-boundary value problem

$$\begin{aligned} u_{tt} - u_{xx} &= 0, \qquad 0 < x < \pi \text{ and } t > 0 \\ u(x, 0) = 1 & \qquad u_t(x, 0) = 0 \quad \text{for } 0 < x < \pi \\ u(0, t) = 0 & \qquad u(\pi, t) = 0 \quad \text{for } t \geq 0. \end{aligned}$$

Ex. 5. Solve the Cauchy problem for the wave equation $u_{tt} - u_{xx} = 0$ with the Cauchy data $u(x, 0) = 0$ and $u_t(x, 0) = \cos x$. Answer: $u(x, t) = \cos x \sin t$.

Ex. 6. Solve the Cauchy problem for the wave equation $u_{tt} - 16u_{xx} = 0$ with the Cauchy data $u(x, 0) = 6 \sin^2 x$ and $u_t(x, 0) = \cos(6x)$.

Ex. 7. Assume that $u(x, t)$ is a solution of the wave equation $u_{tt} - u_{xx} = 0$ for $x \in (0, 1)$ and $t > 0$ with the Cauchy data $u(0, t) = 0 = u(1, t)$ for $t > 0$, $u(x, 0) = f(x)$ and $u_t(x, 0) = g(x)$ for $x \in (0, 1)$. Consider the energy integral

$$E(t) := \int_0^1 [(u_x(x, t))^2 + (u_t(x, t))^2]\, dx.$$

Show that the energy is conserved, i.e. $E(t) = E(0)$ for $t \geq 0$. Use this to show that the Cauchy problem for the wave equation is well-posed and that it admits a unique solution.

Ex. 8. Solve the nonhomogeneous wave equation

$$\begin{aligned} u_{tt} - u_{xx} &= f(x, t) \qquad (1) \\ u(x, 0) = g(x) & \qquad u_t(x, 0) = h(x). \end{aligned}$$

Hint: Use Green's (divergence) theorem to evaluate $\int\int(u_{tt} - u_{xx})\,dx\,dt$ over the region bounded by the characteristics through the point (x,t) and the line segment between the points $(x-t,0)$ and $(x+t,0)$. Answer is given by d'Alembert's formula:

$$\begin{aligned} u(x,t) &= \frac{1}{2}[g(x+t) - g(x-t)] + \frac{1}{2}\int_{x-t}^{x+t} h(\sigma)\,d\sigma \\ &\quad + \frac{1}{2}\int_0^t \int_{x-(t-\tau)}^{x+(t-\tau)} F(\sigma,\tau)\,d\sigma d\tau. \end{aligned}$$

Ex. 9 (Duhamel's Principle)**.** Let $U(x,t,s)$ be C^2 in x and t ad continuous in s and be a solution of

$$\begin{aligned} U_{tt} - c^2 U_{xx} &= 0 \quad \text{for } x \in \mathbb{R}, t > s \geq 0 \\ U(x,0,s) &= 0 \quad \text{for } x \in \mathbb{R}, s \geq 0 \\ U_t(x,0,s) &= f(x,s) \quad \text{for } x \in \mathbb{R}, s \geq 0. \end{aligned}$$

Let $v(x,t)$ be defined by

$$v(x,t) := \int_0^t U(x,t-s,s)\,ds.$$

Then

$$u(x,t) := v(x,t) + \frac{1}{2}[g(x+t) - g(x-t)] + \frac{1}{2c}\int_{x-t}^{x+t} h(\sigma)\,d\sigma$$

is the solution of the nonhomogeneous problem of the wave equation.