Higher Dimensional Wave Equation

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We let $r := d(x, 0) \equiv (\sum_{j=1}^{n} x_j^2)^{1/2}$. Recall the polar coordinates on $\mathbb{R}^n \setminus \{0\}$:

$$x = r(x)\xi, \quad \text{where } \xi := \frac{x}{\|x\|}$$

so that the Lebesgue measure is given by

$$dx_1 \cdots dx_n = r^{n-1} dr dS(\xi)$$

where $dS(\xi)$ stands for the surface measure on the unit sphere $S = \{x \in \mathbb{R}^n : \|x\| = 1\}$.

Ex. 1. Assume that u is a C^2 function on which is *radial*, that is, $u(x) = u(\|x\|)$ for all $x \in \mathbb{R}^n$. Show that

$$\Delta u := \sum_{j=1}^{n} \frac{\partial^2 u}{\partial x_j^2} = u_{rr} + \frac{n-1}{r} u_r. \tag{1}$$

Ex. 2. Assume that $u(x, t)$ is radial in $x \in \mathbb{R}^3$ and solves the wave equation $u_{tt} - \Delta u = 0$. Show that $v(r, t) := ru(x, t)$ solves the one dimensional wave equation $v_{tt} - v_{rr} = 0$.

Ex. 3. Let h be a continuous function on $\mathbb{R}^n$. The *spherical means* $M_h(x, r)$ for $x \in \mathbb{R}^n$ and $r > 0$ is defined by

$$M_h(x, r) := \frac{1}{\omega_n} \int_{|\xi|=1} h(x + r\xi) \, dS(\xi) \tag{2}$$

where $dS(\xi)$ is the surface measure (or area element) of the unit sphere $\{\xi : |\xi| = 1\}$ and ω_n is the total area of the unit sphere. Establish the Darboux equation

$$\left(\frac{\partial^2}{\partial r^2} + \frac{n-1}{r} \frac{\partial}{\partial r} \right) M_h(x, r) = \Delta M_h(x, r). \tag{3}$$

Ex. 4. Explain Poisson's method of spherical means to reduce the Cauchy problem for the wave equation a PDE (called the Euler-Poisson-Darboux equation) in two variables r and t:

$$\frac{\partial^2}{\partial t^2} M_u(x, r, t) = c^2 \left(\frac{\partial^2}{\partial r^2} + \frac{n-1}{r} \frac{\partial}{\partial r} \right) M_u(x, r, t). \tag{4}$$

What are the relevant initial conditions for this equation? How do you recover $u(x, t)$?

Ex. 5. Derive the Kirchoff's formula for the solution of the Cauchy problem for the wave equation three dimensions:

$$u(x,t) = \frac{1}{4\pi}\frac{\partial}{\partial t}\left(t\int_{|\xi|=1} g(x+ct\xi)\,dS(\xi)\right) + \frac{t}{4\pi}\int_{|\xi|=1} h(x+ct\xi)\,dS(\xi). \tag{5}$$

Ex. 6. Analyze the solution obtained in the last exercise. More precisely, introduce the terms domain of dependence and the range of influence and identify them.

Ex. 7. Explain Hadamard's method of descent to solve the Cauchy problem for the wave equation in two dimensions. Identify the domain of dependence and the range of influence in this case.

Ex. 8. Compare and contrast the domains of dependences, the range of influences of the solutions of the Cauchy problems for the wave equations in dimensions $n = 1, 2, 3$. What is known as Huygen's principle?

Ex. 9. Define the energy integral $E(t)$ of a function $u \in C^1(\mathbb{R}^n \times R)$.

State and prove the conservation of energy for the Cauchy problem of the wave equation for Cauchy data with compact supports.

Ex. 10. Use the energy integral to prove that the uniqueness of solutions of the Cauchy problem for the wave equation.

Establish the well-posedness of the Cauchy problem for the wave equation.

Ex. 11 (Local Energy Integral)**.** Let u be a C^2 solution of the Cauchy problem of the wave equation with the initial data $u(x,0) = g(x)$ and $u_t(x,0) = h(x)$. For $x_0 \in \mathbb{R}^n$ and $t_0 > 0$, if $g = 0 = h$ on $C = \{x \in \mathbb{R}^n : |x - x_0| \leq ct_0\}$, then $u(x_0,t_0) = 0$. (This is a hard exercise. See the hint below.)

How does this proof differ from the uniqueness proof of Ex. 10?

Ex. 12 (Duhamel's Principle)**.** Let $w \in C^{[n/2]+1}(\mathbb{R}^n \times \mathbb{R})$. For each $s > 0$, let $v(x,t;s)$ be the solution of $v_{tt} - \Delta v = 0$ with the initial conditions $v(x,0;s) = 0$ and $v_t(x,0;s) = w(s,x)$. Then $u(x,t) := \int_0^t v(x,t-s;s)\,ds$ satisfies the inhomogeneous wave equation $u_{tt} - u_{xx} = w$with 0 Cauchy data $u(x,0) = 0$ and $u_t(x,0) = 0$.

Ex. 13. Solve the inhomogeneous wave equation with an arbitrary Cauchy data.

Meta Hint: Except the first two problems, all are essentially 'book-work' from §3.2—§3.3 of McOwen's *Partial Differential Equations.*