Some Results about Heat Equation

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1 Fundamental Solution of the Heat Operator

Definition 1. The function

$$E(x,t) = \begin{cases} \frac{1}{(2a\sqrt{\pi t})^n} \exp(-\frac{|x|^2}{4a^2 t}) & \text{for } t > 0 \\ 0 & \text{for } t \leq 0 \end{cases}$$

is a fundamental solution of the Heat operator $\frac{\partial}{\partial t} - a^2 \Delta$.

Recall that this means that for each $\varphi \in C_c^\infty(\mathbb{R}^n \times \mathbb{R})$, we have

$$\int_{\mathbb{R}^{n+1}} E(x,t) \left(-\frac{\partial \varphi}{\partial t} - \Delta\varphi \right)(x,t) = \varphi(0,0).$$

The function E is locally integrable in $\mathbb{R}^{n+1}$ since $E = 0$ for $t \leq 0$ and for $t > 0$ we have

$$\int_{\mathbb{R}^n} E(x,t)\, dx = \frac{1}{(2a\sqrt{\pi t})^n} \int_{\mathbb{R}^n} \exp(-\frac{|x|^2}{4a^2 t})\, dx = \prod_{j=1}^{n} \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-\xi^2}\, d\xi = 1. \tag{1}$$

For $t > 0$ and $x \in \mathbb{R}^n$, $E(x,t)$ is smooth. We therefore compute

$$\frac{\partial E}{\partial t} = \left(\frac{|x|^2}{4a^2 t^2} - \frac{n}{2t} \right) E \tag{2}$$

$$\frac{\partial E}{\partial x_j} = -\frac{x_j}{2a^2 t} E \tag{3}$$

$$\frac{\partial^2 E}{\partial x_j^2} = \left(\frac{x_j^2}{4a^2 t^2} - \frac{1}{2a^2 t} \right) E. \tag{4}$$

It follows from (2) and (4 that $E(x,t)$ satisfies the heat equation for $t > 0$:

$$\frac{\partial}{\partial t} E - a^2 \Delta E = 0. \tag{5}$$

Let $\varphi \in C_c^\infty(\mathbb{R}^{n+1})$. We now compute:

$$\begin{aligned}
(\frac{\partial E}{\partial t} - a^2\Delta E, \varphi) &= -(E, \frac{\partial \varphi}{\partial t}) - (E, a^2\Delta\varphi) \\
&= -\int_{\mathbb{R}^n}\int_0^\infty E(x,t)\left(\frac{\partial \varphi}{\partial t} + a^2\Delta\varphi\right) dxdt \\
&= -\lim_{\varepsilon\to 0}\int_{\mathbb{R}^n}\int_\varepsilon^\infty E(x,t)\left(\frac{\partial \varphi}{\partial t} + a^2\Delta\varphi\right) dxdt \\
&= \lim_{\varepsilon\to 0}\left[\int_{\mathbb{R}^n} E(x,\varepsilon)\varphi(x,\varepsilon)\, dx \right. \\
&\quad \left. + \int_\varepsilon^\infty\int_{\mathbb{R}^n}\left(\frac{\partial E}{\partial t} - a^2\Delta E\right)\varphi\, dxdt\right] \\
&\quad \text{(by integration by parts)} \\
&= \lim_{\varepsilon\to 0}\int_{\mathbb{R}^n} E(x,\varepsilon)\varphi(x,\varepsilon)\, dx. \qquad (6)
\end{aligned}$$

To obtain the last equality, we used (5). We change the variables in (6). Put $y = \frac{x}{2a\sqrt{\varepsilon}}$. Then the integral on the right side of (6) becomes

$$\int_{\mathbb{R}^n} \pi^{-n/2}\exp(-|y|^2)\varphi(2a\sqrt{\varepsilon}, \varepsilon)\, dy.$$

In this integral, the function $e^{-|y|^2}$ is in $L^1(\mathbb{R}^n)$ and φ is bounded on $\mathbb{R}^n$ so that we can apply the dominated convergence theorem to conclude that

$$\int_{\mathbb{R}^n} \pi^{-n/2}\exp(-|y|^2)\varphi(2a\sqrt{\varepsilon}, \varepsilon)\, dy \to \pi^{-n/2}\int_{\mathbb{R}^n}\exp(-|y|^2)\varphi(0,0)\, dy = \varphi(0,0).$$

□

2 Maximum Principle for Bounded Domains

Let Ω be a bounded domain. Let $\Omega_T = \Omega \times (0,T)$, for $T \in \mathbb{R}_+$. Then the *parabolic boundary* $\partial_p\Omega_T$ of Ω_T is defined by

$$\partial_p\Omega_T := \partial\Omega_T \setminus (\Omega \times \{T\}).$$

Theorem 2 (Maximum principle for bounded domains)**.** *Let $u \in C^{2,1}(\Omega_T) \cap C(\overline{\Omega}0_T)$. Assume that $u_t - \Delta u \le 0$ in Ω_T. Then*

$$\sup_{\Omega_T} u = \sup_{\partial_p\Omega_T} u. \qquad (7)$$

Proof. Let $\varepsilon > 0$ be arbitrary but fixed. Consider the function

$$v(x,t) := u(x,t) - \varepsilon t, \qquad (x,t) \in \overline{\Omega}_{T-\varepsilon}.$$

It satisfies

$$v_t - \Delta v \le -\varepsilon < 0 \qquad \text{in } \Omega_{T_\varepsilon}. \qquad (8)$$

Since v is continuous in $\overline{\Omega}_{T-\varepsilon}$, it achieves its maximum at some $(x_0, t_0) \in \overline{\Omega}_{T_\varepsilon}$. If $(x_0, t_0) \notin \partial_p \Omega_{T-\varepsilon}$, by elementary calculus, $(v_t - \Delta v)(x_0, t_0) \geq 0$, contradicting (8). Therefore conclude that $(x_0, t_0) \in \partial_p \Omega_{T_\varepsilon}$. For each $\varepsilon \in (0, 1)$, we have

$$u(x,t) \leq 2\varepsilon T + \sup_{\partial_p \Omega_T} u, \qquad \text{for } (x,t) \in \overline{\Omega}_{T-\varepsilon}.$$

□

Ex. 3. Let $u \in C^{2,1}(\Omega_T) \cap C(\overline{\Omega}0_T)$. Assume that $u_t - \Delta u \geq 0$ in Ω_T. Then

$$\inf_{\Omega_T} u = \inf_{\partial_p \Omega_T} u. \tag{9}$$

Corollary 4. *Let $u \in C^{2,1}(\Omega_T) \cap C(\overline{\Omega}0_T)$. Assume that $u_t - \Delta u = 0$ in Ω_T. Then*

$$\| u \|_{\infty, \Omega_T} = \| u \|_{\infty, \partial_p \Omega_T}. \tag{10}$$

Remark 5. Theorem 2 is a weak maximum principle since it does not exclude the possibility of u attaining its supremum at some other points in $\overline{\Omega}_T$. For example, u could be identically constant in Ω_T. A strong maximum principle will assert that this is the only other possibility.

Corollary 6 (Uniquenss for bounded domains)**.** *There exists at most one solution of the boundary value problem $u \in C^{2,1}(\Omega_T) \times C(\overline{\Omega}_T)$ with $u_t - \Delta u = f$ in Ω_T for $f \in C(\Omega)$ and $u = g$ on $\partial_p \Omega_T$ for $g \in C(\partial_p \Omega_T)$.*

Proof. If both u and v are solutions, then $w = u - v$ is a solution of $w_t - \Delta w = 0$ and $w = 0$ on $\partial_p \Omega_T$. Hence $w \equiv 0$ by Theorem 2. □

Remark 7. A boundary value problem for the heat equation with data prescribed on the whole boundary $\partial \Omega_T$ is in general not well-posed. For example, consider the domain $R = (0,1) \times (0,1)$ and a function $\varphi \in C(\partial R)$ which takes an absolute maximum on the open line segment $\{0 < x < 1\} \times \{t = 1\}$, then the problem $u_t - \Delta u = 0$ with the boundary condition $u = \varphi$ on ∂R cannot have a solution for it would violate Theorem 2.

3 Maximum Principle on $\mathbb{R}^n$

Let $\Omega_T := \mathbb{R}^n \times (0, T)$. A point of Ω_T is denoted by (x, t). By $C^{2,1}(\Omega_T)$, we mean the collection of functions on Ω_T that are C^2 in the $x \in \mathbb{R}^n$-variable and C^1 in the t variable.

Theorem 8 (Maximum Principle for $\mathbb{R}^n \times (0,T)$)**.** *Let $u \in C^{2,1}(\Omega_T) \cap C(\overline{\Omega}_T)$. Let $u_t - \Delta u \leq 0$ in Ω_T and $u(x, 0) = g(x)$ with $g \in C(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$. Assume that u satisfies the following growth condition: There exist constants $C, \alpha, r \in (0, \infty)$ such that*

$$u(x,t) \leq C e^{\alpha \| x \|^2}, \qquad \text{for } (x,t) \in \Omega_T \text{ and } |x| > r. \tag{11}$$

Then $\sup_{\Omega_T} u \leq \sup_{\mathbb{R}^n} g$.

Proof. Assume that T is so small that $4\alpha T < 1$. Let $U := B(0,\rho) \times (0,T)$. Consider the function

$$v := u - \frac{\varepsilon}{[4\pi(T-t)]^{n/2}} e^{|x|^2/4(T-t)}.$$

One easily checks that $v_t - \Delta v \leq 0$ in U. We can therefore apply the maximum principle for bounded domains. Let $\partial_p U$ denote the parabolic boundary of U. We have

$$\begin{aligned} v(x,0) &\leq \sup_{\mathbb{R}^n} u(x,0) = \sup_{\mathbb{R}^n} g(x) \\ v(x,t)_{|x|=\rho} &\leq Ce^{\alpha\rho^2} - \varepsilon(4\pi T)^{-n/2} e^{\rho^2/4T}. \end{aligned} \tag{12}$$

Since $\alpha < 4T$, it follows that

$$v \leq 0 \quad \text{on } |x| = \rho. \tag{13}$$

Since ε is arbitrary, the result follows from (12) and (13) *provided that* $4\alpha T < 1$.

If $4\alpha T \geq 1$, subdivide the strip in the t-variable into finitely many strips of width less than $1/4\alpha$. □

Theorem 9 (Uniqueness for the Cauchy problem)**.** *Let* $g \in C(\mathbb{R}^n)$ *and* $f \in C(\mathbb{R}^n \times [0,T))$. *Then there exists at most one solution* $u \in C^{2,1}(\mathbb{R}^n \times (0,T) \cap C(\mathbb{R}^n \times [0,T)$ *of the Cauchy problem*

$$\begin{aligned} u_t - \Delta u &= f \quad \text{in } \mathbb{R}^n \times (0,T) \\ u &= g \quad \text{on } \mathbb{R}^n \times \{t=0\} \end{aligned}$$

satisfying the growth estimate $|u(x,t)| \leq Ae^{\alpha|x|^2}$ *for all* x *with* $|x| > r$ *some constants* C, α, r.

Proof. If u and v both are solutions of the Cauchy problem, apply the last theorem to the function $w = \pm(u - v)$. □

4 Example of Non-uniqueness

Proposition 10. *There exist nonidentically zero solutions of the Cauchy problem* $u_t - \Delta u = 0$ *in* $\mathbb{R} \times (0,\infty)$ *and* $u(x,0) = 0$.

Proof. For $z \in \mathbb{C}$, let

$$\varphi(z) := \begin{cases} e^{-1/z^2}, & z \neq 0 \\ 0, & z = 0. \end{cases}$$

Define

$$u(x,t) = \begin{cases} \sum_{n=0}^{\infty} \frac{d^n}{dt^n}\varphi(t)\frac{x^{2n}}{(2n)!}, & t > 0 \\ 0 & t = 0. \end{cases} \tag{14}$$

We proceed formally.

$$\lim_{t\to 0} u(x,t) = \sum_{n=0}^{\infty} \frac{d^n}{dt^n}\varphi(0)\frac{x^{2n}}{(2n)!} = 0 \tag{15}$$

$$\begin{aligned}
\frac{\partial^2 u}{\partial x^2} &= \sum_{n=0}^{\infty} \frac{d^n}{dt^n}\varphi(t)2n(2n-1)\frac{x^{2n}}{(2n)!} \\
&= \sum_{n=1}^{\infty} \frac{d^n}{dt^n}\varphi(t)\frac{x^{2(n-1)}}{(2(n-1))!} \\
&= \sum_{n=0}^{\infty} \frac{d^{n+1}}{dt^{n+1}}\varphi(t)\frac{x^{2n}}{(2n)!} \qquad (16)\\
&= \frac{\partial u}{\partial t}. \qquad (17)
\end{aligned}$$

These calculations are rigorous in view of the following lemma. □

Lemma 11. *The series in (14 - 16) are uniformly convergent on compact subsets of* $\mathbb{R}\times\mathbb{R}_+$.

Proof. The function $z \mapsto \varphi(z)$ is holomorphic in $\mathbb{C}\setminus\{0\}$.We identify the t-axis with the real axis of the complex plane. If $t > 0$ is fixed, the circle $\gamma(\theta) := t + \frac{t}{2}e^{i\theta}$ does not meet the origin. By Cauchy integral formula we have

$$\frac{d^n}{dt^n}\varphi(t) = \frac{n!}{2\pi i}\int_\gamma \frac{\varphi(z)}{(z-t)^{n+1}}dz.$$

It follows from this that

$$|\frac{d^n}{dt^n}\varphi(t)| \le \frac{n!}{2\pi}(\frac{2}{t})^n \int_0^{2\pi} e^{-\Re(z^{-2})}d\theta.$$

For z on γ, we have $z^2 = t^2(1+\frac{1}{2}e^{i\theta})^2$. So,

$$z^{-2} = t^{-2}\frac{(1+\frac{1}{4}e^{-2i\theta}+e^{-i\theta})}{|(1+\frac{1}{2}e^{i\theta})^2|^2}.$$

From this, we get $\Re(z^{-2}) \ge (2t)^{-2}$, and hence

$$|\frac{d^n}{dt^n}\varphi(t)| \le n!(\frac{2}{t})^n e^{-1/4t^2}.$$

Fix $x\in\mathbb{R}$ and $R>0$. Then for all x with $|x|<R$, using the Stirling's inequality

$$\frac{2^n(n!)^2}{(2n)!} \le 1,$$

the series in (14) is seen to be majorized, term by term, by the uniformly convergent series

$$e^{-1/4t^2}\sum_{n=0}^{\infty}(\frac{1}{t})^n\frac{(R^2)^n}{n!} = e^{-1/4t^2}e^{R^2/t}.$$

□

5 Backward Uniqueness

Theorem 12 (Backward Uniqueness)**.** *Assume that u and v are solutions of the Cauchy problem for the heat equation in $U_T = U \times (0,T)$ with the same boundary condition $u = v = 0$ on $\partial U \times [0,T]$. If $u(x,T) = v(x,T)$ for $x \in U$, then $u = v$ on U_T.*

Proof. Write $w = u = v$. Let

$$E(t) := \int_U w^2(x,t)\,dx, \quad t \in [0,T].$$

By hypothesis $E(T) = 0$ and we may assume that $E(t) > 0$ for $0 \le t < T$. We compute the first derivative of E.

$$\begin{aligned} E'(t) &= 2\int_U w w_t\,dx \\ &= 2\int_U w\Delta w\,dx \\ &= -2\int_U |\nabla w|^2 dx \le 0. \end{aligned} \tag{18}$$

Furthermore, we compute the second derivative of E.

$$\begin{aligned} E''(t) &= -4\int_U \sum_{k=1}^n w_{x_k} w_{x_k t}\,dx \\ &= 4\int_U \Delta w w_t\,dx \\ &= 4\int_U (\Delta w)^2\,dx. \end{aligned} \tag{19}$$

Now since $w = 0$ on ∂U,

$$\begin{aligned} \int_U |\nabla w|^2 dx &= -\int_U w\Delta w dx \\ &\le \left(\int_U w^2\,dx\right)^{1/2} \left(\int_U (\Delta w)^2\,dx\right)^{1/2}. \end{aligned}$$

Thus (18) and (19) imply

$$\begin{aligned} (E'(t))^2 &= 4\left(\int_U |Dw|^2\,dx\right)^2 \\ &\le \left(\int_U w^2\,dx\right)\left(4\int_U (\Delta w)^2\,dx\right)^{1/2} \\ &= E(t)E''(t). \end{aligned}$$

That is,

$$E''(t)E(t) \ge (E'(t))^2. \tag{20}$$

Let $f(t) := \log E(t)$. Then $f''(t) \geq 0$ by (20) and hence is convex on the interval $(0, T)$. Consequently, if $\lambda, t \in (0,1)$, we have

$$f(\lambda t) = f((1-\lambda)0 + \lambda t) \leq (1-\lambda)f(0) + \lambda f(t).$$

It follows that

$$E(\lambda t) \leq E(0)^{1-\lambda} E(t)^{\lambda}, t \in (0,T),$$

and so,

$$E(\lambda T) \leq E(0)^{1-\lambda} E(T)^{\lambda}.$$

Since $E(T) = 0$, this entails in $E(t) = 0$ for $t \in (0,T)$, a contradiction to our assumption. □

6 Nonhomogeneous Equation — Duhamel's Principle

Theorem 13 (Duhamel). *Let $g \in C(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$. Let f be a function on $\mathbb{R}^n \times [0,T]$ such that f is $C^{2,1}$ and f_t, $\partial_j f$ and $\partial_j \partial_k f$ exist, are continuous and bounded on $\mathbb{R}^n \times [0,T]$. Define*

$$u(x,t) := \int_{\mathbb{R}^n} K(x-y,t) g(y)\, dy + \int_0^t \int_{\mathbb{R}^n} K(x-y, t-s) f(y,s)\, dy\, ds. \qquad (21)$$

Then u is bounded, $u \in C^{2,1}(\mathbb{R}^n \times [0,T])$ and is a solution of the initial value problem $u_t - \Delta u = f$ and $u(x,0) = g(x)$ for $x \in \mathbb{R}^n$ and $0 \leq t \leq T$.

Proof. We shall only sketch the main step. It suffices to show that the second term of (21) is a solution of $u_t - \Delta u = f$. We compute

$$\begin{aligned}
\frac{\partial}{\partial t} \int_0^t \int_{\mathbb{R}^n} K(x-y,t-s) f(y,s)\, dy\, ds &= \int_0^t \int_{\mathbb{R}^n} \frac{\partial}{\partial t} K(x-y,t-s) f(y,s)\, dy\, ds \\
&\quad \lim_{s \to t} \int_{\mathbb{R}^n} K(x-y,t-s) f(y,s)\, dy\, ds \\
&= \int_0^t \int_{\mathbb{R}^n} \Delta K(x-y,t-s) f(y,s)\, dy\, ds + f(x,t) \\
&= \Delta \left(\int_0^t \int_{\mathbb{R}^n} K(x-y,t-s) f(y,s)\, dy\, ds \right) \\
&\quad + f(x,t).
\end{aligned}$$

□