Every compact metric space is a continuous image of the cantor set

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Let C be the Cantor set of middle thirds:

$$C := \{x \in \mathbb{R} : x = \sum_k a_k/3^k, \text{ where } a_k \text{ is either 0 or 2}\}.$$

The theorem of the title says that any compact metric space is a continuous image of the Cantor set.

We let $I^{\mathbb{N}}$ denote the product of countably infinite product of the unit interval $[0,1]$.

Lemma 1. *If (X,d) is a compact metric space, then X is homeomorphic to a (necessarily closed) subset of $I^{\mathbb{N}}$.*

Proof. We may and do assume that the metric on X is bounded by 1. (Justify this.) Since X is a compact metric space, there exists a countable dense subset, say $\{x_n : n \in \mathbb{N}\}$. We define $F\colon X \to I^N$ by setting

$$F(x) := (d(x,x_1), d(x,x_2), \dots, d(x,x_n), \dots).$$

The coordinate functions $\pi_n \circ F\colon X \to [0,1]$ are continuous. By the universal mapping property of the product topology, the function F is continuous. We claim that F is one-one. Suppose that $x, y \in X$ are such that $F(x) = F(y)$. Since $\{x_n\}$ is dense in X, there exists a sequence (x_{n_k}) such that $x_{n_k} \to x$ as $k \to \infty$. Hence $d(x_{n_k}, x) \to 0$ as $k \to \infty$. Since $F(x) = F(y)$, it follows that $d(x,x_n) = d(y,x_n)$ for all n. In particular, $d(y,x_{n_k}) = d(x,x_{n_k}) \to 0$. Since the limit of a sequence in a metric space is unique, we deduce that $x = y$. This establishes our claim. Since X is compact and I^∞ is Hausdorff, it follows that $F\colon X \to F(X)$ is a homeomorphism. □

Lemma 2. *The unit interval $[0,1]$ is the continuous image of the Cantor set.*

Proof. Easy. Consider the map $g\colon C \to [0,1]$ given by $g(\sum_k a_k/3^k) = \sum_k a_k/2^{k+1}$. □

Lemma 3. *The Cantor set C is homeomorphic to $\prod_{\mathbb{N}}\{0,2\}$, the countable product of the two point space $\{0,2\}$ with discrete topology.*

Proof. Consider $h(\sum_k a_k/3^k) = (a_1, a_2, \ldots)$. □

Lemma 4. *The Cantor set is homeomorphic to the countable product of Cantor sets.*

Proof. Observe that $\mathbb{N}$ can be written as a countable union of (infinitely) countable subsets. It therefore follows that $\prod_1^\infty \{0,1\}$ is homeomorphic to the countable product of spaces each of which is a countable product of two point spaces. The result follows from the last lemma. □

Lemma 5. *The Hilbert cube I^∞ is the continuus image of the cantor set.*

Proof. In view of Lemma 4, we may assume that any $x \in C$ is of the form $(x_1, x_2, \ldots)$, $x_i \in C$. We define $G(x) = (g(x_1), \ldots, g(x_n), \ldots)$, where g is as in the proof of Lemma 2. □

Lemma 6. *If K is a closed subset of the Cantor set C, then K is the continuous image of the Cantor set.*

Proof. Let the middle-two-thirds set C' be the set of real numbers of the form $\sum_k b_k/6^k$ where b_k is either 0 or 5. The obvious, as seen in Lemma 3, it is homeomorphic to $\prod_1^\infty \{0,1\}$. Hence the cantor set C and the middle-two-thirds set C' are homeomorphic.

The set C' has the property that if $x, y \in C'$, then thier mid point $(x+y)/2$ does not lie in C'. Now assume that K' is a closed subset of C'. If $x' \in C$,, then there exists a unique point $k_x \in K'$ such that $d(x', k_x) = d(x', K')$. The function $k\colon C' \to K$" given by $k(x) = k_x$ is a continuous, onto retraction. □

We can now complete the proof of the theorem. Let us assume that the given compact metric space X is a subset of $I^{\mathbb{N}}$. Let F be a continuous function from the Cantor set C onto $I^{\mathbb{N}}$. Then $F^{1}(X)$ is a closed subset of C and it is mapped by F onto X.

References:

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