Maximum Principle Heat Equation

S. Kumaresan
School of Math. and Stat.
University of Hyderabad
Hyderabad 500046
kumaresa@gmail.com

1 Maximum Principle for Bounded Domains

Let Ω be a bounded domain. Let $\Omega_T = \Omega \times (0, T)$, for $T \in \mathbb{R}_+$. Then the *parabolic boundary* $\partial_p \Omega_T$ of Ω_T is defined by

$$\partial_p \Omega_T := \partial \Omega_T \setminus (\Omega \times \{T\}).$$

Theorem 1 (Maximum principle for bounded domains). *Let $u \in C^{2,1}(\Omega_T) \cap C(\overline{\Omega}0_T)$. Assume that $u_t - \Delta u \leq 0$ in Ω_T. Then*

$$\sup_{\Omega_T} u = \sup_{\partial_p \Omega_T} u. \tag{1}$$

Proof. Let $\varepsilon > 0$ be arbitrary but fixed. Consider the function

$$v(x,t) := u(x,t) - \varepsilon t, \qquad (x,t) \in \overline{\Omega}_{T-\varepsilon}.$$

It satisfies

$$v_t - \Delta v \leq -\varepsilon < 0 \qquad \text{in } \Omega_{T_\varepsilon}. \tag{2}$$

Since v is continuous in $\overline{\Omega}_{T-\varepsilon}$, it achieves its maximum at some $(x_0, t_0) \in \overline{\Omega}_{T_\varepsilon}$. If $(x_0, t_0) \notin \partial_p \Omega_{T-\varepsilon}$, by elementary calculus, $(v_t - \Delta v)(x_0, t_0) \geq 0$, contradicting (2). Therefore conclude that $(x_0, t_0) \in \partial_p \Omega_{T_\varepsilon}$. For each $\varepsilon \in (0, 1)$, we have

$$u(x,t) \leq 2\varepsilon T + \sup_{\partial_p \Omega_T} u, \qquad \text{for } (x,t) \in \overline{\Omega}_{T-\varepsilon}.$$

□

Ex. 2. Let $u \in C^{2,1}(\Omega_T) \cap C(\overline{\Omega}0_T)$. Assume that $u_t - \Delta u \geq 0$ in Ω_T. Then

$$\inf_{\Omega_T} u = \inf_{\partial_p \Omega_T} u. \tag{3}$$

Corollary 3. *Let $u \in C^{2,1}(\Omega_T) \cap C(\overline{\Omega}0_T)$. Assume that $u_t - \Delta u = 0$ in Ω_T. Then*

$$\|u\|_{\infty, \Omega_T} = \|u\|_{\infty, \partial_p \Omega_T}. \tag{4}$$

Remark 4. Theorem 1 is a weak maximum principle since it does not exclude the possibility of u attaining its supremum at some other points in $\overline{\Omega}_T$. For example, u could be identically constant in Ω_T. A strong maximum principle will assert that this is the only other possibility.

Corollary 5 (Uniquenss for bounded domains)**.** *There exists at most one solution of the boundary value problem* $u \in C^{2,1}(\Omega_T) \times C(\overline{\Omega}_T)$ *with* $u_t - \Delta u = f$ *in* Ω_T *for* $f \in C(\Omega)$ *and* $u = g$ *on* $\partial_p \Omega_T$ *for* $g \in C(\partial_p \Omega_T)$*.*

Proof. If both u and v are solutions, then $w = u - v$ is a solution of $w_t - \Delta w = 0$ and $w = 0$ on $\partial_p \Omega_T$. Hence $w \equiv 0$ by Theorem 1. □

Remark 6. A boundary value problem for the heat equation with data prescribed on the whole boundary $\partial \Omega_T$ is in general not well-posed. For example, consider the domain $R = (0,1) \times (0,1)$ and a function $\varphi \in C(\partial R)$ which takes an absolute maximum on the open line segment $\{0 < x < 1\} \times \{t = 1\}$, then the problem $u_t - \Delta u = 0$ with the boundary condition $u = \varphi$ on ∂R cannot have a solution for it would violate Theorem 1.

2 Maximum Principle on $\mathbb{R}^n$

Let $\Omega_T := \mathbb{R}^n \times (0, T)$. A point of Ω_T is denoted by (x, t). By $C^{2,1}(\Omega_T)$, we mean the collection of functions on Ω_T that are C^2 in the $x \in \mathbb{R}^n$-variable and C^1 in the t variable.

Theorem 7 (Maximum Principle for $\mathbb{R}^n \times (0,T)$)**.** *Let* $u \in C^{2,1}(\Omega_T) \cap C(\overline{\Omega}_T)$*. Let* $u_t - \Delta u \leq 0$ *in* Ω_T *and* $u(x, 0) = g(x)$ *with* $g \in C(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$*. Assume that* u *satisfies the following growth condition: There exist constants* $C, \alpha, r \in (0, \infty)$ *such that*

$$u(x,t) \leq C e^{\alpha \|x\|^2}, \qquad \text{for } (x,t) \in \Omega_T \text{ and } |x| > r. \tag{5}$$

Then $\sup_{\Omega_T} u \leq \sup_{\mathbb{R}^n} g$*.*

Proof. Assume that T is so small that $4\alpha T < 1$. Let $U := B(0, \rho) \times (0, T)$. Consider the function

$$v := u - \frac{\varepsilon}{[4\pi(T-t)]^{n/2}} e^{|x|^2/4(T-t)}.$$

One easily checks that $v_t - \Delta v \leq 0$ in U. We can therefore apply the maximum principle for bounded domains. Let $\partial_p U$ denote the parabolic boundary of U. We have

$$\begin{aligned} v(x,0) &\leq \sup_{\mathbb{R}^n} u(x,0) = \sup_{\mathbb{R}^n} g(x) \\ v(x,t)_{|x|=\rho} &\leq C e^{\alpha \rho^2} - \varepsilon (4\pi T)^{-n/2} e^{\rho^2/4T}. \end{aligned} \tag{6}$$

Since $\alpha < 4T$, it follows that

$$v \leq 0 \quad \text{on } |x| = \rho. \tag{7}$$

Since ε is arbitrary, the result follows from (6) and (7) *provided that* $4\alpha T < 1$.

If $4\alpha T \geq 1$, subdivide the strip in the t-variable into finitely many strips of width less than $1/4\alpha$. □

Theorem 8 (Uniqueness for the Cauchy problem). *Let $g \in C(\mathbb{R}^n)$ and $f \in C(\mathbb{R}^n \times [0,T))$. Then there exists at most one solution $u \in C^{2,1}(\mathbb{R}^n \times (0,T) \cap C(\mathbb{R}^n \times [0,T)$ of the Cauchy problem*

$$\begin{aligned} u_t - \Delta u &= f \quad \text{in } \mathbb{R}^n \times (0,T) \\ u &= g \quad \text{on } \mathbb{R}^n \times \{t=0\} \end{aligned}$$

satisfying the growth estimate $|u(x,t)| \le Ae^{\alpha|x|^2}$ for all x with $|x| > r$ some constants C, α, r.

Proof. If u and v both are solutions of the Cauchy problem, apply the last theorem to the function $w = \pm(u - v)$. □

3 Example of Non-uniqueness

Proposition 9. *There exist nonidentically zero solutions of the Cauchy problem $u_t - \Delta u = 0$ in $\mathbb{R} \times (0, \infty)$ and $u(x,0) = 0$.*

Proof. For $z \in \mathbb{C}$, let

$$\varphi(z) := \begin{cases} e^{-1/z^2}, & z \neq 0 \\ 0, & z = 0. \end{cases}$$

Define

$$u(x,t) = \begin{cases} \sum_{n=0}^{\infty} \frac{d^n}{dt^n}\varphi(t)\frac{x^{2n}}{(2n)!}, & t > 0 \\ 0 & t = 0. \end{cases} \tag{8}$$

We proceed formally.

$$\lim_{t\to 0} u(x,t) = \sum_{n=0}^{\infty} \frac{d^n}{dt^n}\varphi(0)\frac{x^{2n}}{(2n)!} = 0 \tag{9}$$

$$\begin{aligned} \frac{\partial^2 u}{\partial x^2} &= \sum_{n=0}^{\infty} \frac{d^n}{dt^n}\varphi(t)2n(2n-1)\frac{x^{2n}}{(2n)!} \\ &= \sum_{n=1}^{\infty} \frac{d^n}{dt^n}\varphi(t)\frac{x^{2(n-1)}}{(2(n-1))!} \\ &= \sum_{n=0}^{\infty} \frac{d^{n+1}}{dt^{n+1}}\varphi(t)\frac{x^{2n}}{(2n)!} \qquad (10) \\ &= \frac{\partial u}{\partial t}. \qquad (11) \end{aligned}$$

These calculations are rigorous in view of the following lemma. □

Lemma 10. *The series in (8 - 10) are uniformly convergent on compact subsets of $\mathbb{R} \times \mathbb{R}_+$.*

Proof. The function $z \mapsto \varphi(z)$ is holomorphic in $\mathbb{C} \setminus \{0\}$.We identify the t-axis with the real axis of the complex plane. If $t > 0$ is fixed, the circle $\gamma(\theta) := t + \frac{t}{2}e^{i\theta}$ does not meet the origin. By Cauchy integral formula we have

$$\frac{d^n}{dt^n}\varphi(t) = \frac{n!}{2\pi i}\int_\gamma \frac{\varphi(z)}{(z-t)^{n+1}}dz.$$

It follows from this that

$$|\frac{d^n}{dt^n}\varphi(t)| \leq \frac{n!}{2\pi}(\frac{2}{t})^n \int_0^{2\pi} e^{-\Re(z^{-2})}d\theta.$$

For z on γ, we have $z^2 = t^2(1 + \frac{1}{2}e^{i\theta})^2$. So,

$$z^{-2} = t^{-2}\frac{(1 + \frac{1}{4}e^{-2i\theta} + e^{-i\theta})}{|(1 + \frac{1}{2}e^{i\theta})^2|^2}.$$

From this, we get $\Re(z{-}2) \geq (2t)^{-2}$, and hence

$$|\frac{d^n}{dt^n}\varphi(t)| \leq n!(\frac{2}{t})^n e^{-1/4t^2}.$$

Fix $x \in \mathbb{R}$ and $R > 0$. Then for all x with $|x| < R$, using the Stirling's inequality

$$\frac{2^n(n!)^2}{(2n)!} \leq 1,$$

the series in (8) is seen to be majorized, term by term, by the uniformly convergent series

$$e^{-1/4t^2}\sum_{n=0}^{\infty}(\frac{1}{t})^n\frac{(R^2)^n}{n!} = e^{-1/4t^2}e^{R^2/t}.$$

□

4 Backward Uniqueness

Theorem 11 (Backward Uniqueness)**.** *Assume that u and v are solutions of the Cauchy problem for the heat equation in $U_T = U \times (0,T)$ with the same boundary condition $u = v = 0$ on $\partial U \times [0,T]$. If $u(x,T) = v(x,T)$ for $x \in U$, then $u = v$ on U_T.*

Proof. Write $w = u = v$. Let

$$E(t) := \int_U w^2(x,t)\,dx, \quad t \in [0,T].$$

By hypothesis $E(T) = 0$ and we may assume that $E(t) > 0$ for $0 \leq t < T$. We compute the first derivative of E.

$$\begin{aligned} E'(t) &= 2\int_U ww_t\,dx \\ &= 2\int_U w\Delta w\,dx \\ &= -2\int_U |\nabla w|^2 dx \leq 0. \end{aligned} \tag{12}$$

Furthermore, we compute the second derivative of E.

$$\begin{aligned} E''(t) &= -4\int_U \sum_{k=1}^n w_{x_k} w_{x_k t}\, dx \\ &= 4\int_U \Delta w w_t \, dx \\ &= 4\int_U (\Delta w)^2 \, dx. \end{aligned} \tag{13}$$

Now since $w = 0$ on ∂U,

$$\begin{aligned} \int_U |\nabla w|^2 dx &= -\int_U w\Delta w dx \\ &\leq \left(\int_U w^2 \, dx\right)^{1/2} \left(\int_U \Delta w^2 \, dx\right)^{1/2}. \end{aligned}$$

Thus (12) and (13) imply

$$\begin{aligned} (E'(t))^2 &= 4\left(\int_U |Dw|^2 \, dx\right)^2 \\ &\leq \left(\int_U w^2 \, dx\right)\left(4\int_U (\Delta w)^2 \, dx\right)^{1/2} \\ &= E(t)E''(t). \end{aligned}$$

That is,

$$E''(t)E(t) \geq (E'(t))^2. \tag{14}$$

Let $f(t) := \log E(t)$. Then $f''(t) \geq 0$ by (14 and hence is convex on the interval $(0, T)$. Consequently, if $\lambda, t \in (0, 1)$, we have

$$f(\lambda t) = f((1-\lambda)0 + \lambda t) \leq (1-\lambda)f(0) + \lambda f(t).$$

It follows that

$$E(\lambda t) \leq E(0)^{1-\lambda} E(t)^\lambda, t \in (0, T),$$

and so,

$$E(\lambda T) \leq E(0)^{1-\lambda} E(T)^\lambda.$$

Since $E(T) = 0$, this entails in $E(t) = 0$ for $t \in (0, T)$, a contradiction to our assumption. □

5 Nonhomogeneous Equation — Duhamel's Principle

Theorem 12 (Duhamel)**.** *Let $g \in C(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$. Let f be a function on $\mathbb{R}^n \times [0, T]$ such that f is $C^{2,1}$ and f_t, $\partial_j f$ and $\partial_j \partial_k f$ exist, are continuous and bounded on $\mathbb{R}^n \times [0, T]$. Define*

$$u(x, t) := \int_{\mathbb{R}^n} K(x-y, t)g(y)\, dy + \int_0^t \int_{\mathbb{R}^n} K(x-y, t-s) f(y, s)\, dy\, ds. \tag{15}$$

Then u is bounded, $u \in C^{2,1}(\mathbb{R}^n \times [0, T])$ and is a solution of the initial value problem $u_t - \Delta u = f$ and $u(x, 0) = g(x)$ for $x \in \mathbb{R}^n$ and $0 \leq t \leq T$.

Proof. We shall only sketch the main step. It suffices to show that the second term of (15) is a solution of $u_t - \Delta u = f$. We compute

$$\begin{aligned}
\frac{\partial}{\partial t}\int_0^t \int_{\mathbb{R}^n} K(x-y,t-s)f(y,s)\,dy\,ds &= \int_0^t \int_{\mathbb{R}^n} \frac{\partial}{\partial t}K(x-y,t-s)f(y,s)\,dy\,ds \\
&\quad \lim_{s\to t}\int_{\mathbb{R}^n} K(x-y,t-s)f(y,s)\,dy\,ds \\
&= \int_0^t \int_{\mathbb{R}^n} \Delta K(x-y,t-s)f(y,s)\,dy\,ds + f(x,t) \\
&= \Delta\left(\int_0^t \int_{\mathbb{R}^n} K(x-y,t-s)f(y,s)\,dy\,ds\right) \\
&\quad + f(x,t).
\end{aligned}$$

□