Little Picard Theorem

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Theorem 1 (Little Picard's Theorem)**.** *Any entire non constant function on $\mathbb{C}$ can miss at the most one point of $\mathbb{C}$.*

Proof of this theorem is a beautiful application of the Uniformization Theorem and various other results from Complex analysis and Algebraic Topology. This shows how various results from different branches come together to give a simple proof of non trivial results.

First of all let us start with the Uniformization theorem. We will not prove this theorem. The proof is quite technical and involved. For a proof, we refer the reader to books on Riemann surfaces.

Theorem 2 (Uniformization Theorem)**.** *Let X be a simply connected Riemann Surface. Then X is biholomorphic to precisely one of the following spaces:*

1) $\mathbb{P}^1(\mathbb{C})$

2) $\mathbb{C}$

3) $B(0,1)$*, the open unit disk in* $\mathbb{C}$*.* □

We now prove the main result.

Proof. Let $f \in H(\mathbb{C})$ miss two points, say, p and q. That is, $f : \mathbb{C} \to \mathbb{C} \setminus \{p, q\}$ is onto. Since $\mathbb{C} \setminus \{p, q\}$ is open in $\mathbb{C}$, it is a Riemann surface. This is not simply connected because we can find a loop which winds p but not null homotopic. Hence there exists a simply connected Riemann surface X which is a universal cover for X. Let $p : X \to \mathbb{C} \setminus \{p, q\}$ be the covering map. Then there exists a lift of f. That is, there exists a holomorphic map $g : \mathbb{C} \to X$ such that $p \circ g = f$. Let Γ be the set of all deck transformations of this covering map p. Then Γ act on X totally properly discontinuously and hence the quotient of X by Γ is $\mathbb{C} \setminus \{p, q\}$. By the Uniformization theorem X is either $\mathbb{P}^1(\mathbb{C})$, $\mathbb{C}$ or $B(0, 1)$.

We claim that X is $B(0, 1)$. Granting the claim for a moment, we complete the proof. The lift g of f is a constant by Liouville's theorem. Hence $f = p \circ g$ is constant. Thus if an entire function misses two points, then it is a constant.

We now prove the claim by contradiction. If $X \neq B(0, 1)$, then either $X = \mathbb{P}^1(\mathbb{C})$ or $X = \mathbb{C}$. We show that both these are untenable.

Let, if possible, $X = \mathbb{P}^1(\mathbb{C})$. Since $\mathbb{P}^1(\mathbb{C})$ is compact, $p(X) = \mathbb{C} \setminus \{p, q\}$ is compact, an absurdity.

Assume that $X = \mathbb{C}$. Since Γ is the group of deck transformation, it is discrete. The set of biholomorphic automorphism of $\mathbb{C}$ is the group $G = \{f_{a,b} : a \in \mathbb{C}^*, b \in \mathbb{C}\}$, where $f_{a,b}(z) = az + b$. The group Γ of deck transformation is a discrete subgroup of G. It is well-known that the only (closed) discrete subgroups of $\mathbb{R}^n$ are of the form $\mathbb{Z}\alpha_1 \oplus \ldots \oplus \mathbb{Z}\alpha_r$, $\alpha_1, \ldots \alpha_r \in \mathbb{R}^n$ are linearly independent over $\mathbb{R}$. (Exercise.) In this case the only possibilities are $\Gamma \simeq \mathbb{Z}$ or $\Gamma \simeq \mathbb{Z} \oplus \mathbb{Z}$.

Therefore, either (i) $X/\Gamma \simeq \mathbb{C}/\mathbb{Z} \simeq \mathbb{C} \setminus \{p, q\}$ or (ii) $X/\Gamma \simeq \mathbb{C}/(\mathbb{Z} \oplus \mathbb{Z}) \simeq \mathbb{C} \setminus \{p, q\}$. In the first case, $\mathbb{C}/\mathbb{Z}$ is homeomorphic to a cylinder. Hence its fundamental group is $\mathbb{Z}$ whereas that of $\mathbb{C} \setminus \{p, q\}$ is a free group on two generators. In the second case, $\mathbb{C}/(\mathbb{Z} \oplus \mathbb{Z})$ is homeomorphic to a torus and hence is compact whereas $\mathbb{C} \setminus \{p, q\}$ is not. □

We would like to comment on the classical proof. One uses the so-called modular function h in that approach. The modular function actually gives a covering map of the upper half-plane $\mathbb{H} := \{z \in \mathbb{C} : \operatorname{Im} z > 0\}$ onto $\mathbb{C} \setminus \{0, 1\}$. Now if $f \colon \mathbb{C} \to \mathbb{C} \setminus \{0, 1\}$ is holomorphic, we get a lift $g \colon \mathbb{C} \to \mathbb{H}$ of f: $f = h \circ g$. Since $\mathbb{H}$ and $B(0, 1)$ are biholomorphic, say, via φ, we see that the holomorphic function $\varphi \circ g$ is a constant by Liouville's theorem. Hence $g = \varphi^{-1} \circ (\varphi \circ g)$ is a constant. Consequently, $f = h \circ g$ is a constant. If $f \colon \mathbb{C} \to \mathbb{C} \setminus \{p, q\}$, then we use the fact $\mathbb{C} \setminus \{p, q\}$ is biholomorphic to $\mathbb{C} \setminus \{0, 1\}$ to reduce it to the earlier case.

Acknowledgement: This was written up and typed by Ajit Kumar. I still remember three of my students who were pursuing Ph.D. across the globe came together to attend this lecture. (Of course, they were vacationing in Bombay!) Many of their friends also turned up from the institutes of Mumbai (then known as Bombay).