Plane Geometries

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Abstract

In this article, we take a pedestrian approach and introduce the reader to the three but rich plane geometries.

The title plane geometries should conjure up the image of points, lines and the incidence relations between them in the mind of the reader. Like all concepts in modern mathematics, we start with a simple set of axioms which are abstractions of the ideal requirements which our objects should have. We start with a (non-empty set) X whose elements are called *points* and a class Ł of special subsets of X to be called *lines* and the *incidence relation*, viz., a point x is *incident on* $\ell \in$ Ł if $x \in \ell$. We may express this last incidence relation by saying that ℓ is incident on x or l passes through x. Now we impose some "natural conditions" on this pair $(X,$ Ł$)$. We require that any two points determine a unique line. This means that given $x, y \in X$ with $x \neq y$ there exists a unique element $l \in$ Ł such that $x, y \in l$. We also demand that any two distinct lines pass through at most one point. That is, given $l, l' \in$ Ł the intersection $l \cap l'$ has at most one point in common. These requirements are enough for the time-being. Any pair $(X,$ Ł$)$ which satisfies the above is called a plane geometry. Let us look at some examples of this concept.

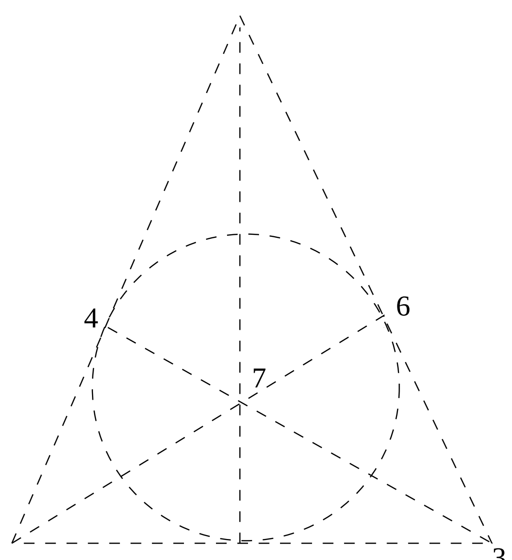
As first example let me give you an uninspiring example. Let $X = \{x, y\}$ any two element set and Ł $:= \{X\}$. Then it satisfies all our requirements and hence is a plane geometry. This shows that we must impose perhaps some other condition so that our plane geometry will be "rich". One such condition may be to require that X has at least 3 elements. Here after we shall assume that our plane geometry satisfies these conditions.

Before the reader gets all knotted up, let me give an example which is the one closest to his intuition and most "well understood". As $(X,$ Ł$)$ take the "Euclidean plane " along with the lines in the plane, that is **the** plane which reader has learnt in his high school. For the more pedantic readers, X is $\mathbb{R}^2$ and Ł is the cosets of one dimensional subspaces of $\mathbb{R}^2$. Hey, do not lose heart if you did not understand any bit of the last line. This article was written for people like you and not for the "pundits!" You can still go ahead and get your thrill!

A second example is again an abstract one to test your staying power. Take a set of 7 elements, say, $X = \{1, 2, \ldots, 7\}$. As lines let me take Ł $:= \{l_i : 1 \leq i \leq 7\}$ where l_i are defined as follows:

$$l_1 = \{1,7,5\} \quad l_2 = \{1,6,3\} \quad l_3 = \{1,4,2\} \quad l_4 = \{2,7,6\}$$
$$l_5 = \{2,5,3\} \quad l_6 = \{3,7,4\} \quad l_7 = \{4,5,6\}$$

You may be tempted to say that this is the reason you never liked mathematics, I share your views and sympathise with you. There is a perfectly geometric way in which the mathematicians visualise this plane. Look at the picture below and ponder over it. The lines as drawn over there are there just for "ornamental sake" to aid our imagination.



To the cognoscenti, this is nothing but the projective plane over the Galois field of 2 elements.

After this esoteric example, let us look at a very concrete plane geometry. This time X is the upper half plane in $\mathbb{R}^2$, that is, $X = \{(x, y) \in \mathbb{R}^2 : y > 0\}$. As lines, we take the collection $\mathcal{V}$ of vertical lines (in the usual Euclidean sense) and $\mathcal{C}$ the collection of semicircles of all possible radii with centres on the x-axis. Thus,

$$\begin{aligned} \mathcal{V} &= \{\ell : \ell := \{(a, y) : a \in \mathbb{R}, y > 0\}\} \\ \mathcal{C} &= \{C_{a,R} := (x, y) : y > 0(x - a)^2 + y^2 = R^2, a \in \mathbb{R}, R > 0\}. \end{aligned}$$

Thus $Ł = \mathcal{V} \cup \mathcal{C}$. Note that the centre of $C_{a,R}$ is not a point of X! I invite the reader to check (at least convince himself using his coordinate geometry) that $(X, Ł)$ is a plane geometry. To check your understanding, answer this question: What is the 'unique' line joining the following pair of points: (a) $p = (1, 1)$ and $q = (-1, 1)$,
(b) $p = (0, 1)$ and $q = (0, 2)$?

As a final "example" (note the quotes), we take X to be the (surface of the) sphere S of unit radius with centre at the origin in $\mathbb{R}^3$. That is, $S := \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 = 1\}$. Here as the set $\tilde{Ł}$ of lines L we take all the great circles L got as the intersection of a plane passing through the origin with S. In particular, given any two vectors (which are not multiples of each other) in S, we take the plane (the two dimensional vector subspace) spanned by them and take its intersection with S. Thus given p and $q(\neq \pm p)$, we have a unique 'line' joining them. But the shrewd reader must have observed that if we take p and $q = -p$, then there are far too many lines (namely great circles) passing through the points p and $-p$. Thus $(S, \tilde{Ł})$ is not a plane geometry according to our definition.

Modern mathematicians do not sit and cry over this kind of set-up. They are crafty and get around this in an ingenious way. We pretend that we cannot distinguish between p and $-p$. That is, as far as we are concerned, p and $-p$ are one and the same point. Thus we take X to be the set of points $[p, -p]$: $X := \{[p, -p]; p \in S\}$.

Those of you who have learnt about equivalence relation will realize that X is the quotient set of S with respect to the equivalence relation $x \equiv y$ iff $x = \pm y$ for $x, y \in S$.

We take Ł to be the image of any L (defined above) in X: Thus,

$$Ł = \{[L] : [L] = \{[x, -x] : x \in L\}\}.$$

I leave the very illuminating task of verifying that $(X, Ł)$ is a plane geometry. If you have difficulty in carrying out this detail, you may still proceed and come back later (after seeing something below connected with this example).

Now if you ask me to give a geometrically visualisable picture (in $\mathbb{R}^2$ or $\mathbb{R}^3$), you have me there. In such cases, what a mathematicians does is this: if he wants to say something about X then he looks its analogue at S, does it there and comes back to X. This principle is illustrated in a paragraph below.

Having done these, you might wonder what all these things lead to? I try to answer this in an oblique way in the rest of this article.

We agree to say that two lines l and l' in a plane geometry are *parallel* if either $l = l'$ or $l \cap l' = \emptyset$. Armed with this definition, let us look at some examples of parallel lines in the above three examples.

In the usual plane $X = \mathbb{R}^2$, if l is the x-axis, then parallel lines are given by $l' = \{(x, y) : y = \text{ constant}\}$.

In the second examples where X is the "upper half plane", if I take l to be the vertical line $\{(0, y) : y > 0\}$, then any line l' parallel to l is given by $l' = \{(a, y) : y > 0\}$ where $a \in \mathbb{R}$ is chosen arbitrarily. Can you find what are the lines parallel to C_a where C_a is defined as above? If you cannot solve this, do not despair, it is kind of solved below.

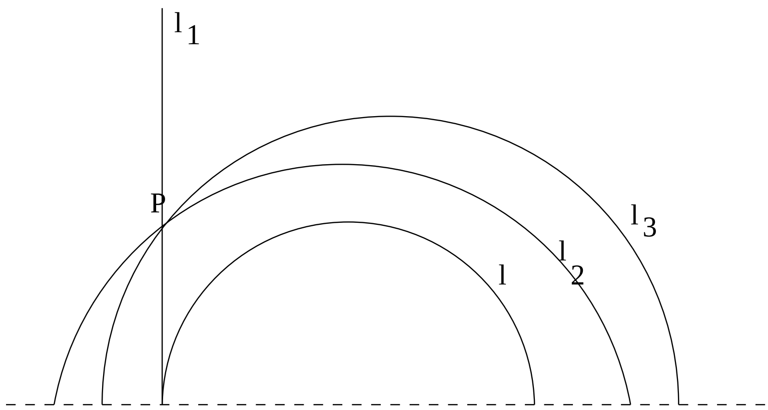
In the third example, l' is a parallel to l iff $l = l'$. That is to say that there are no nontrivial pair of parallel lines in this plane geometry! To verify this, let us look at the 'lines' in S. Given any two lines L and L' such that $L = P \cap S$ and $L' = P' \cap S$ for some planes P and P' through the origin. Since P and P' are planes through the origin, they intersect along a (usual Euclidean) line through the origin. This line in turn intersects the sphere S at two antipodal points x and $-x$. Thus the images l of L and l' of L' intersect at the point $[x, -x] \in X$. Thus any two lines intersect in this plane geometry!

Just as an aside, we invite the reader to check that the esoteric example of a plane consisting of 7 points and 7 lines also has this property.

All these examples should inevitably goad us into thinking of the (controversial) Euclid's parallel postulate. Euclid, in his definition of plane geometry has postulated that $(X, Ł)$ has the property that given a line l and a point p not on it, there exists a unique line l' passing through p parallel to l.

Let us see what happens in our three examples. In the usual plane, his postulate is true. In fact, if we develop this via linear algebra his postulate becomes a theorem. Those of you who know about cosets of subspaces of a vector space should know that either any two cosets are the same or they do not have any intersection.

In the second example, something fantastic happens. Take as l any semicircle with the centre on the x-axis and a point $p = (p_1, p_2) \in X \subset \mathbb{R}^2$ not on the line l as in the picture. Take as $l' = \{(p_1, y) : y > 0\}$. Then the line l' is parallel to l. (That they seem to have a point on the x-axis in common is irrelevant as x-axis is not part of our plane X.) You can also convince yourself that you can draw an infinite number of semicircles passing through p which are parallel to l.



In the third example, given l and a point $p \notin l$ as we have seen earlier it is not possible to find a line l' parallel to l and passing through p.

Thus our three planes exhibit all possible variations of the Euclid's parallel postulate:

1. In the usual plane, given a line and a point p not on the line there exist a **unique** line l' passing though p and parallel to l. Any plane geometry having this property is called a **Euclidean geometry**.

2. In the second example (where X is the upper half plane), given a line l and a point p not on it, there exist **infinitely many** lines l' parallel to the given line passing through the point p. A plane having this behavior is known as a **hyperbolic plane**.

3. In the last example (in which X is the sphere), given a line l and a point p not on it, there exists **no** line through p which is parallel to l. A plane exhibiting this phenomenon is known as an **elliptic plane**.

The last two are hence known as **Non-Euclidean Geometries**.

Now you might wish to raise the following point: "You started with some arbitrary X and took some special class of curves as the lines in X. So anything can happen. What is the big idea?" Well, you are correct. But I have a way to justify what I did above and which also explains why I chose these special curves as the lines.

If you agree with me that the intuitive notion of a line is that it is in some vague sense the "shortest" curve joining "nearby" points on it, then one can show that the curves are indeed lines in their respective planes, provided a proper interpretation of length of curves is given. One imitates the formula for arc-length in Euclidean geometry to define the length of a curve c as follows:

$$\text{length of } c := \int_a^b (\dot{c}\cdot\dot{c})^{1/2}\varphi(c(t))\,dt.$$

Here $\varphi\colon X \to \mathbb{R}^+$ is continuous function. Thus the tangent vector $\dot{c}(t)$ has length **not** the usual Euclidean length $(\dot{c}\cdot\dot{c})^{1/2}$ but $\varphi(c(t))(\dot{c}\cdot\dot{c})^{1/2}$. Thus the length of the tangent vector has a magnification factor depending upon its position. This kind of thing occurs "naturally". If you have any doubt, ask any physicist about Lorentz metric and Minkowski space-time etc.

Finally two teasers: If $X = \mathbb{R}^3$, and lines are the usual lines as in three dimensional coordinate geometry, then $\mathbb{R}^3$ is also a "plane geometry" according to our definition. But a plane must surely be "2-dimensional" and $\mathbb{R}^3$ is 3-dimensional. Is there any further condition to be imposed so that $\mathbb{R}^3$ is disqualified from being a "plane"?

The second one is as follows: Consider $\mathbb{R}^3$. As X we take all the standard lines L in $\mathbb{R}^3$ passing through the origin: $\mathcal{L} := \{t(x,y,z) : t \in \mathbb{R}\}$ for a fixed non-zero $(x,y,z) \in \mathbb{R}^3$. As for planes we take the standard planes P in $\mathbb{R}^3$ through the origin: $P := \{(x,y,z) \in \mathbb{R}^3 : ax+by+cz=0\}$ for a fixed non-zero $(a,b,c) \in \mathbb{R}^3$.

Then $(X,\mathcal{L})$ is a plane geometry whose points are "standard lines" in $\mathbb{R}^3$ passing through the origin and whose lines are "standard planes" in $\mathbb{R}^3$ passing through the origin in $\mathbb{R}^3$. This is essentially the third example (involving S). In algebraic language, this is known as the projective plane over $\mathbb{R}$.

A Defense: The purpose of this article is to excite you and lure you into geometry. Hence I have taken care not to smother you with most precise statements which may leave you cold. I crave the indulgence of those of you who are sticklers for details and of those who find the agony and the history of the geometry missing in this write-up. I find that books whose titles include the phrase "non-Euclidean geometry" go into all kinds of logical knots which leave a student desirous of knowing about these objects, in despair.

Unfortunately, I do not have any specific reference for the material above. I advise you to contact your friendly neighbourhood geometer or *me*. As a general reading to whet your appetite for geometry I may suggest Coxeter's *Introduction to Geometry*.

Added on 28 June 2012:

A good reference to all the above in a more rigorous manner is the following:

S Kumaresan and G Santhanam, *An Expedition to Geometry*, Hindustan Book Agency, Delhi, 2005.