An Example of a Prime Ideal

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Consider the ring $R := \mathbb{Q}[x, y]$ and the ideal $I :=< x - y^2 >$. We claim that the ideal I is prime. We prove this by showing that the quotient ring R/I is an integral domain, namely, the polynomial ring $Q[t]$.

Let $\varphi\colon R \to Q[t]$ be the ring homomorphism given by $\varphi(x) = t^2$ and $\varphi(y) = t$. Clearly, $I \subset \ker \varphi$. We shall show that $\ker \varphi = I$.

Let $S := R/I$. We claim that every element of S can be written in the form $p(x)+q(x)y+I$, for $p, q \in \mathbb{Q}[x]$. To prove this, consider an arbitrary coset $f(x, y) + I \in S$. Then we can write this as

$$
\begin{aligned}
f(x, y) + I &= q(x) + \text{ terms with odd powers of } y + \text{ terms with even powers of } y + I \\
&= q(x) + y \cdot \text{ terms with even powers of } y + \text{ terms with even powers of } y + I.
\end{aligned}
$$

Since any term with an even power of y is of the form $g(x)y^{2k}$ and since $x + I = y^2 + I$, we see that we can replace third and fourth terms above by terms of the form $h_1(x)y + I$ and $h_2(x) + I$. Hence the claim follows.

We are now ready for the kill. Let $f(x, y) \in \ker \varphi$. Using the claim, we write $f(x, y) = g(x) + h(x)y + \psi(x, y)$ where $\psi \in I$. Now, we operate φ on both sides of the equation to get

$$\varphi(f)(t) = g(t^2) + h(t^2) \cdot t = 0.$$

Noting that there are no common powers of t in the two terms $g(t^2)$ and $h(t^2) \cdot t$ we see that the coefficients of g and h must be zero. Hence $f = \psi \in I$.