A topological Proof of the Infinitutde of Primes

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Theorem 1. *The set of primes $\mathbb{P}$ is infinite.*

Proof. For $a \in \mathbb{Z}$ and $b \in \mathbb{N}$ consider $P_{a,b} := \{a + kb : k \in \mathbb{Z}\}$. We say that $U \subset \mathbb{Z}$ is open if either it is empty or if for each $a \in U$, there exists $b \in \mathbb{N}$ such that $P_{a,b} \subset U$. One checks that this defines a topology on $\mathbb{P}$. Note that any nonempty open set is infinite.

Each $P_{a,b}$ is open since for each $x \in P_{a,b}$, we have $P_{x,b} = P_{a,b}$.

Each $P_{a,b}$ is closed. For,

$$\begin{aligned} x \notin P_{a,b} &\iff x \neq a + kb \text{ for any } k \in \mathbb{Z} \\ &\iff x - a \text{ is not a multiple of } b \\ &\iff x - a = i + kb \text{ for some } 1 \leq i \leq b - 1 \text{ by Division algorithm.} \end{aligned}$$

Thus $\mathbb{Z} \setminus P_{a,b} = \cup_{i=1}^{n} P_{a+i,b}$, union of open sets. Hence $P_{a,b}$ is closed.

Now, given any $x \neq \pm 1$, there exists a prime p which divides x. Hence $x = kp$ for some $k \in \mathbb{Z}$. Thus, $\mathbb{Z} \setminus \{\pm 1\} = \cup_{p \in \mathbb{P}} P_{0,p}$. If $\mathbb{P}$ were finite, then $\mathbb{Z} \setminus \{\pm 1\}$, being a finite union of closed sets would be closed. But then the finite set $\{\pm 1\}$ is open, a contradiction. $\square$