Topics in Measure Theory

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1 Product Measure and Fubini Theorem

Let $(X, \mathcal{A}, \mu)$ and $(Y, \mathcal{B}, \nu)$ be two measure spaces. We wish to find a 'natural' σ-algebra $\mathcal{C}$ on $X \times Y$ and a measure α on $\mathcal{C}$. The obvious requirements on $\mathcal{C}$ and α is that if $A \in A$ and $B \in \mathcal{B}$, then $A \times B \in \mathcal{C}$ and that $\alpha(A \times B) = \mu(A) \times \nu(B)$. Since we "know" how to integrate with respect to μ and ν, the next requirement is that if $f \colon (X \times Y, \mathcal{C}) \to [0, \infty]$ is measurable, then

$$\int_{X \times Y} f(x, y)\, d\alpha = \int_X \left(\int_Y f(x, y)\, d\mu \right) d\nu = \int_Y \left(\int_X f(x, y)\, d\nu \right) d\mu. \tag{1}$$

Note that if we take $R := A \times B$ with $A \in \mathcal{A}$ and $B \in \mathcal{B}$, then $f := \chi_R \equiv \mathbf{1}_R$, the characteristic (or the indicator) function of R, satisfies (1). Note also that for (1) to make sense, we need to ensure that the functions $x \to f(x, y)$ for y fixed and $y \to f(x, y)$ for x fixed are measurable on the respective spaces. One may hope to extend the result in (1) to finite linear combinations of such $\mathbf{1}_R$'s and use monotone convergence theorem to extend it to all non-negative measurable functions.

So, one way of finding $\mathcal{C}$ and α could be to use (1). Let $\mathcal{R}$ denote the class of rectangles $A \times B$ with $A \in \mathcal{A}$ and $B \in \mathcal{B}$. Note that if we wish to have $\mathcal{R} \subset \mathcal{C}$, we need to ensure that $(X \times Y) \setminus (A \times B) \in \mathcal{C}$. It is eay to verify that the complement of such a rectangle can be written as

$$(X \times Y) \setminus (A \times B) = ((X \setminus A) \times B) \cup (X \times (Y \setminus B)\,,$$

that is, the complement of a rectangle is a finite disjoint union of rectangles. This suggests that we should consider the collection $\mathcal{S}$ of all such finite disjoint unions of rectangles. It turns out $\mathcal{S}$ is an algebra of sets on $X \times Y$ according to the next definition.

Definition 1. Let $\mathcal{A} \subset P(X)$ be a class of subsets of a set X. We say that $\mathcal{A}$ is an *algebra of sets* if (1) $X \in \mathcal{A}$, (2) if $A \in \mathcal{A}$, then $X \setminus A \in A$ and (3) whenever $A, B \in \mathcal{A}$, their union $A \cup B \in \mathcal{A}$.

Since we plan to use monotone convergence theorem, it is necessary that the σ-algebra $\mathcal{C}$ shoudl contain all subsets which are obtainable by taking limts of monotone sequences in $\mathcal{S}$. This motivates the next definition.

Definition 2. Let $\mathcal{M} \subset P(X)$ be a class of subsets of a set X. We say that $\mathcal{A}$ is a *monotone class of sets* (i) if $A_n \in \mathcal{M}$ with $A_n \uparrow A$, then $A \in \mathcal{M}$ and (ii) if $B_n \in \mathcal{M}$ with $B_n \downarrow B$, then $B \in \mathcal{M}$.

The crucial tool in this article is the following lemma.

Lemma 3. *If $\mathcal{A}$ is an algebra of subsets of X and $\mathcal{M}$ is the smallest monotone class containing $\mathcal{A}$, then $\mathcal{M}$ is a σ-algebra.*

Proof. We claim that $\mathcal{M}$ is closed under complementation. Let $E \in M$. Consider the class $\mathcal{M}_0 := \{E \in \mathcal{M} : X \setminus E \in \mathcal{M}\}$. Since $\mathcal{A}$ is an algebra, $X \setminus A \in \mathcal{A} \subset \mathcal{M}$. Hence $\mathcal{A} \subset M_0$. It is easy to check that $\mathcal{M}_0$ is a monotone class. Hence we conclude that $\mathcal{M} \subset M_0$. Hence the claim follows.

For $A \subset X$, we let $\mathcal{M}_A := \{E \in \mathcal{M} : E \cap A \in \mathcal{M}\}$. Since $\mathcal{A}$ is an algebra, it follows that, **if** $A \in \mathcal{A}$, then $\mathcal{A} \subset \mathcal{M}_A$. It is again an easy observation that $\mathcal{M}_A$ is a monotone class. Consequently, $\mathcal{M} \subset M_A$. That is, if $A \in \mathcal{A}$ and $E \in \mathcal{M}$, then $E \cap A \in \mathcal{M}$.

Let $E \in \mathcal{M}$ be arbitrary. Consider $\mathcal{M}_E$. By the last paragraph, $\mathcal{A} \subset \mathcal{M}_E$. It is easy to see that $\mathcal{M}_E$ is a monotone class and hence $\mathcal{M} \subset \mathcal{M}_E$. That is, finite intersections of elements of $\mathcal{M}$ are again in $\mathcal{M}$. The same is true of union of finite number of elements of $\mathcal{M}$. □

Let $\mathcal{M} \subset P(X \times Y)$ be the smallest monotone class containing the algebra $\mathcal{S}$ of the class of all subsets of $X \times Y$ which can be written as disjoint union of a finite number of 'measurable rectangles' $R = A \times V$ where $A \in \mathcal{A}$ and $B \in \mathcal{B}$. By the monotone class lemma, $\mathcal{M}$ is a σ-algebra on $X \times Y$.

Lemma 4. *Assume that $\mu(X) < \infty$ and $\nu(Y) < \infty$. Let*

$$\mathcal{C} := \left\{ F \subset X \times Y : \int_X \left(\int_Y \mathbf{1}_F(x,y)\, d\mu \right) d\nu = \int_Y \left(\int_X \mathbf{1}_F(x,y)\, d\nu \right) d\mu \right\}.$$

Then $\mathcal{M} \subset \mathcal{C}$.

Proof. The steps are the same. We show that $\mathcal{A} \subset \mathcal{C}$ and then $\mathcal{C}$ is a monotone class. Where do we need the finite measure hypothesis? To deal with the decreasing sequences! □

Theorem 5 (Existence of Product Measure)**.** *Let $(X, \mathcal{A}, \mu)$ and $(Y, \mathcal{B}, \nu)$ be two σ-finite measure spaces. Then there exists a unique measure α on $\mathcal{M}$ with the property that $\alpha(A \times B) = \mu(A)\nu(B)$ for all $A \in \mathcal{A}$ and $B \in \mathcal{B}$.*

Proof. First assume that $\mu(X)$ and $\nu(Y)$ are finite. The last lemma shows that for $E \in \mathcal{M}$, the definition

$$\alpha(E) := \int_X \left(\int_Y \mathbf{1}_F(x,y)\, d\mu \right) d\nu = \int_Y \left(\int_X \mathbf{1}_F(x,y)\, d\nu \right) d\mu.$$

is well-defined. Also, α is finitely additive on $\mathcal{M}$ due to the linearity of the integrals. By the monotone converegence theorem, α is also countably additive. If β is any measure on $\mathcal{M}$

with the required property, then the class of subsets E for which $\alpha(E) = \alpha(E)$ is a monotone class which contain $\mathcal{S}$. Hence $\alpha = \beta$ on $\mathcal{M}$.

In the general case, write $X = \cup_n A_n$ and $Y = \cup_n B_n$ where the unions are disjoint $\mu(A_n) < \infty$ and $\nu(B_n) < \infty$ for $n \in \mathbb{N}$. Given any $E \in \mathcal{M}$, let $E_{mn} := E \cap (A_m \times B_n)$. For each m, n, thanks to the finite case, we have

$$\int_X \int_Y \mathbf{1}_{E_{mn}}\, d\nu\, d\mu = \int_Y \int_X \mathbf{1}_{E_{mn}}\, d\mu\, d\nu.$$

We can sum these terms over m and n (in any order), to get by countable addivity and monotone convergence theorem that

$$\int_X \left(\int_Y \mathbf{1}_F(x,y)\, d\mu \right) d\nu = \int_Y \left(\int_X \mathbf{1}_F(x,y)\, d\nu \right) d\mu, \text{ for } E \in \mathcal{M}.$$

If we set $\alpha(E)$ to be one of these integrals, then by the linearity of the integrals, α is finitely additive. It is countably additive by the monotone convergence theorem. It also has the required property on $\mathcal{R}$.

If β is any other meaure on $\mathcal{M}$ with this property, then writing $E = \cup_{m,n} E_{mn}$ and applying the uniqueness part of the finite measure case to conclude $\alpha(E_{mn}) = \beta(E_{mn})$ for all m, n. Then $\alpha(E) = \sum_{m,n} \alpha(E_{mn}) = \sum_{m,n} \beta(E_{mn}) = \beta(E)$ by counatble additivity. □

Remark 6. The assumption σ-finiteness in the last theorem is essential. For, take $X = Y = [0,1]$ but μ to be counting measure and ν to be the Lebesgue measure. Then the diagonal $D := \{(x,x) : x \in [0,1]\}$ is measurable in the product space with

$$\int_X \int_Y \mathbf{1}_D\, d\nu\, d\mu = 0 \text{ whereas } \int_Y \int_X \mathbf{1}_D\, d\mu\, d\nu = 1.$$

Note that μ is not σ-finite.

Theorem 7 (Fubini-Tonelli)**.** *Let $(X, \mathcal{A}, \mu)$ and $(Y, \mathcal{B}, \nu)$ be two σ-finite measure spaces. Let $f \colon X \times Y \to [0, \infty]$ measurable with respect to product σ-algebra $\mathcal{M}$. Then* (1) *holds true.*

The same conclusion is true if we assume $f \in L^1(X \times Y, \mathcal{M}, \alpha)$. Here $\int f(x,y)\, d\mu(x)$ is defined for almost all y etc.

Proof. For nonnegative simple functions, the result follows from Theorem 5 and the additivity of the integrals. For nonnegative measurable functions, it follows from MCT.

For $f \in L^1$, apply the result ot each of f^+ and f^-. □

The following extension is quite useful in practice.

Theorem 8. *Let $f \colon X \times Y \to \mathbb{R}$ be $\mathcal{M}$-measurable. Assume that one of the iterated integrals of $|f|$ exists. Then*
(i) *$f \in L^1(X \times Y, \mathcal{C}, \alpha)$.*
(ii) *$\int_Y f(x,y)\, d\nu(y)$ is defined for all most all x. If it is defined to be 0 on the set of measure zero, then it is measurable and lies in $L^1(X, \mathcal{A}, \mu)$. The analogous result is also true for $\int_X f(x,y)\, d\mu(x)$.* (iii) *The double integral of f exists and is equalt to the each of the iterated integrals.*

Proof. Apply Fubini-Tonelli to each of f^+ and f^-. $\square$

We now give two typical applications of these results.

Example 9. It is easy to see that if $f, f \in C_c(\mathbb{R})$, then the folwoing definition makes sense:

$$f * g(x) := \int_R f(x-y)g(y)\,dy.$$

It truns out that if $f, g \in L^1(R)$ (with respect to the Lebesgue measure), then also it makes sense almost everywhere. The *only* way to prove this is to show that the integral $\int_R \left(\int_R |f(x-y)g(y)|\,dy\right)\,dx$ exists and appeal to Fubini theorem. The details are left to the reader.

Example 10. Let $P\colon \mathbb{R}^n \to \mathbb{R}$ be a polynomial function. Let $E := \{x \in \mathbb{R}^n : P(x) = 0\}$. Then E is of Lebegue measure zero in $\mathbb{R}^n$. Apply Fubini to $\mathbf{1}_E$ to the n-fold iterated integral and observe that if $x_2 = a_2, \ldots, x_n + a_n$ are held fixed, then the polynomial $x_1 \mapsto P(x_1, a_2 \ldots, a_n)$ has only finitely many zeros and hence $\int_R \mathbf{1}_E(x_1, a_2, \ldots, a_n)\,dx_1 = 0$.

2 Radon-Nikodym and Lebesgue Decompostion Theorems

Theorem 11. *Let μ and ν be two finite measure on a measurable space $(X, \mathcal{B})$. Then*
(i) *there exist measures ν_1 and ν_2 such that $\nu = \nu_1 + \nu_2$ where $\nu_1 \ll mu$ and $\nu_2 \perp \mu$.*
(ii) *There exists a nonnegative measurable function φ on $(X, \mathcal{B})$ such that $\nu_1(E) = \int_X \varphi\,d\mu$ for $E \in \mathcal{B}$.*

Proof. Let $\sigma := \mu + \nu$. Consider $T\colon L^2(X, \mathcal{B}, \sigma) \to \mathbb{R}$ given by $Tf := \int_X f\,d\mu$. Then T is a bounded linear functional. Hence there exists $g \in L^2(\sigma)$ such that $Tf = \int_X fg\,d\mu$. It follows

$$\int f(1-g)\,d\sigma = \int f\,d\mu, \qquad \text{for all } f \in L^2(\sigma). \tag{2}$$

$g \geq 0$ almsot everywhere σ: If false, take $f = \mathbf{1}_E$ where $E := \{g < 0\}$ in (2) to arrive at the contradiction $\sigma(E) + \int_E g\,d\sigma = \mu(E)$.

$g \leq 1$ almsot everywhere σ: If false, take $f = \mathbf{1}_E$ where $E := \{g > 1\}$ in (2) to arrive at the contradiction $\mu(E) < 0$.

Let $A := \{0 \leq g < 1\}$ and $B := \{g = 1\}$.

Take $f = \mathbf{1}_B$ in (2) to deduce at $\mu(B) = 0$. We define

$$\nu_A(E) := \nu(E \cap A) \text{ and } \nu_B(E) := \nu(E \cap B), \text{ for } E \in \mathcal{B}.$$

Then $\mu(B) = 0$ and $\nu_B(X \setminus B) = \nu_B(A) = \nu(A \cap B) = \nu(\emptyset) = 0$. Hence $\mu \perp \nu_B$.

We claim: $\nu_A \ll \mu$: If $\mu(E) = 0$, we want to prove that $\nu_A(E) = \mu(E \cap A) = 0$. So, assume $\mu(E) =$ and $E \subset A$. Taking $f = \mathbf{1}_E$ in (2), we get $\int_E (1-g)\,d\sigma = 0$. Hence $0 = \mu(E) = \sigma(E) = \mu(E) + \nu(E)$ so that $\nu(E) = \nu(E \cap A) = \nu_A(E)$. Hence the claim.

The following proof of Radon-Nikodym theorem is based on the original argument by John von Neumann. We suppose that μ and ν are real, nonnegative, and finite. The extension to the σ-finite case is a standard exercise, as is μ-a.e. uniqueness of Radon-Nikodym derivative. Having done this, the thesis also holds for signed and complex-valued measures.

Let $(X, \mathcal{F})$ be a measurable space and let $\mu, \nu : \mathcal{F} \to [0, R]$ two finite measures on X such that $\nu(A) = 0$ for every $A \in \mathcal{F}$ such that $\mu(A) = 0$. Then $\sigma = \mu + \nu$ is a finite measure on X such that $\sigma(A) = 0$ if and only if $\mu(A) = 0$.

Consider the linear functional $T : L^2(X, \mathcal{F}, \sigma) \to \mathbb{R}$ defined by

$$Tu = \int_X u \, d\mu \;\; \forall u \in L^2(X, \mathcal{F}, \sigma) \, . \tag{3}$$

T is well-defined because μ is finite and dominated by σ, so that $L^2(X, \mathcal{F}, \sigma) \subseteq L^2(X, \mathcal{F}, \mu) \subseteq L^1(X, \mathcal{F}, \mu)$; it is also linear and bounded because $|Tu| \leq \|u\|_{L^2(X,\mathcal{F},\sigma)} \cdot \sqrt{\sigma(X)}$. By Riesz representation theorem, there exists $g \in L^2(X, \mathcal{F}, \sigma)$ such that

$$Tu = \int_X u \, d\mu = \int_X u \cdot g \, d\sigma \tag{4}$$

for every $u \in L^2(X, \mathcal{F}, \sigma)$. Then $\mu(A) = \int_A g \, d\sigma$ for every $A \in \mathcal{F}$, so that $0 < g \leq 1$ μ- and σ-a.e. (Consider the former with $A = \{x \mid g(x) \leq 0\}$ or $A = \{x \mid g(x) > 1\}$.) Moreover, the second equality in (4) holds when $u = \chi_A$ for $A \in \mathcal{F}$, thus also when u is a simple measurable function by linearity of integral, and finally when u is a (μ- and σ-a.e.) nonnegative $\mathcal{F}$-measurable function because of the monotone convergence theorem.

Now, $1/g$ is $\mathcal{F}$-measurable and nonnegative μ- and σ-a.e.; moreover, $\frac{1}{g} \cdot g = 1$ σ- and μ-a.e. Thus, for every $A \in \mathcal{F}$,

$$\int_A \frac{1}{g} \, d\mu = \int_A d\sigma = \sigma(A) \tag{5}$$

Since σ is finite, $1/g \in L^1(X, \mathcal{F}, \mu)$, and so is $f = \frac{1}{g} - 1$. Then for every $A \in \mathcal{F}$

$$\nu(A) = \sigma(A) - \mu(A) = \int_A \left(\frac{1}{g} - 1 \right) d\mu = \int_A f \, d\mu \, .$$

□

3 Complex Measures

Lemma 12. *Let $T := \{z_k \in \mathbb{C} : 1 \leq k \leq N\}$ be a finite subset of complex numbers. Then there exists a subset $S \subset T$ such that*

$$|\sum_{z \in S} z| \geq \frac{1}{\pi} \sum_{k=1}^{N} |z_k|. \tag{6}$$

Proof. For $t \in [0, 2\pi]$, we let $S_t := \{z \in T : -\pi/2 \leq t - \arg(z) \leq \pi/2\}$. (Draw a picture.) Let $f(t) := |\sum_{z \in S_t} z|$. We plan to show that there exists $\theta \in [0, 2\pi]$ such that

$$f(\theta) \geq \frac{1}{\pi} \sum_{k=1}^{N} |z_k|.$$

Note that f is piecewise constant on $[0, 2\pi]$. Observe that

$$\begin{aligned}
|\sum_{z \in S_t} z| &= |e^{-it} \sum_{z \in S_t} z| \\
&= |\sum_{z \in S_t} e^{-it} z| \\
&\geq |\mathrm{Re}\left(\sum_{z \in S_t} e^{-it} z\right)| \\
&= |\sum_{z \in S_t} \mathrm{Re}\,(e^{-it} z)| \\
&= |\sum_{z \in S_t} |z| \cos(t - \arg(z))| \\
&= \sum_{z \in S_t} |z| \cos(t - \arg(z)).
\end{aligned}$$

Hence we obtain

$$\begin{aligned}
\int_0^{2\pi} f(t)\, dt &= \int_0^{2\pi} |\sum_{z \in S_t} z|\, dt \\
&\geq \int_0^{2\pi} \sum_{z \in S_t} |z| \cos(t - \arg(z))\, dt \\
&= \sum_{k=1}^{N} \int_{\arg(z_k) - \pi/2}^{\arg(z_k) + \pi/2} |z| \cos(t - \arg(z))\, dt \\
&= \sum_{k=1}^{N} 2|z_k|.
\end{aligned}$$

The last is due to the fact that $\int_{-\pi/2}^{\pi/2} \cos t\, dt = 2$.

By the extreme value theorem, there exists θ suchthat $f(\theta) \geq f(t)$ for $t \in [0, 2\pi]$. It follows that

$$2\pi f(\theta) \geq \int_0^{2\pi} f(t)\,dt \geq 2\sum_{k=1}^{N} |z_k|.$$

□