Proper Maps

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Definition 1. A map $f\colon X \to Y$ is said to be *proper* if for every compact subset $L \subset Y$, the inverse image $f^{-1}(L)$ is a compact subset of X.

Example 2. Any continuous map from a compact space to any hausdorff space Y is proper.

Example 3. Let p be a nonconstant polynomial with complex coefficients. A most important and typical example of a proper map is the function $z \mapsto p(z)$. Recall the standard estimate: There exists $R > 0$ such that

$$|p(z)| \geq \frac{|a_n|}{2}|z|^n, \text{ for } |z| \geq R, \text{ where } p(z) = \sum_{k=0}^{n} a_k z^k.$$

Let K be a compact subset of $\mathbb{C}$. Since p is continuous, $p^{-1}(K)$ is closed. If $p^{-1}(K)$ is not compact, we conclude that $p^{-1}(K)$ is not bounded. (Why? Heine-Borel theorem!) Hence there exists a sequence $z_n \in p^{-1}(K)$ such that $|z_n| \to \infty$, but $p(z_n) \in K$ for all n. By the estimate quoted above, $p(z_n) \to \infty$. But since $p(z_n) \in K$ and K is compact, $\{p(z_n) : n \in \mathbb{N}\}$ is bounded. This contradiction shows that p is proper.

Ex. 4. The exponential map $\exp\colon \mathbb{R} \to \mathbb{R}$ or $\exp\colon \mathbb{C} \to \mathbb{C}$ is not proper.

Lemma 5. *Let $f\colon X \to Y$ be a closed map. Assume that $f^{-1}(y)$ is compact for each $y \in Y$. Then f is proper.*

Proof. Let $L \subset Y$ be a compact subset. Let $\{U_i : i \in I\}$ is an open cover of $K := f^{-1}(L)$. For each $y \in L$, by hypothesis, $f^{-1}(y)$ is compact. Hence, there exists a finite set $J_y \subset I$ such that $\{U_i : i \in J_y\}$ is a finite subcover of $f^{-1}(y)$. Let $U_y := \cup_{i \in J_y} U_i$. Then U_y is open and so $A_y := X \setminus U_y$ is closed in X. Since f is closed, the set $V_y := Y \setminus f(C_y)$ is open in Y. Note that $f^{-1}(V_y) \subset U_y$. Since $y \in V_y$, the collection $\{V_y : y \in L\}$ is an open cover of the compact set L. Hence there exists a finite number of points y_j, $1 \leq j \leq n$ such that $L \subset V_1 \cup \cdots \cup V_n$ where $V_j := V_{y_j}$. But then

$$\begin{aligned} f^{-1}(L) &\subset f^{-1}(V_1) \cup \cdots \cup f^{-1}(V_n) \\ &\subset U_1 \cup \cdots \cup U_n \\ &= \cup\{U_i : i \in J_{y_i}, 1 \leq i \leq n\}, \end{aligned}$$

a finite subcover. $\square$

Lemma 6. *Let X be compact. Then for any topological space Y, the projection $\pi_Y : X \times Y \to Y$ is closed.*

Proof. Let $L \subset X \times Y$ be closed. We have to show that $\pi_Y(L)$ is closed in Y. We show that its complement is open in Y. Let $y \in Y$ but $y \notin \pi_Y(L)$. Note that this means that $(x, y) \in L$ for any $x \in X$. What we plan to do is something similar to the preliminary step, the so-called tube lemma, in the proof of compactness of $X \times Y$: There exists an open set V such that $y \in V$ and $(x, y') \notin L$ for any $x \in X$ and $y' \in V$. From this it follows that such a $V \subset Y \setminus \pi_Y(L)$.

Since L is closed and $(x, y) \notin L$, we can find a basic open set $U_x \times V_x$ such that $(x, y) \in U_x \times V_x \subset (X \times Y) \setminus L$. By the compactness of X, we can find $x_1, \ldots, x_n \in X$ such that $U_i := U_{x_i}$, $1 \leq i \leq n$, cover X. Let $V := V_1 \cap \cdots \cap V_n$, where, as is our standard practice $V_i := V_{x_i}$, $1 \leq i \leq n$. Note that V is an open set containing y. We have

$$(X \times Y) \cap L = [(U_1 \cup \cdots \cup U_n) \times (V_1 \cap \cdots \cap V_n)] = \emptyset.$$

□

Proposition 7. *If X is compact, then $\pi_Y : X \times Y \to Y$ is proper.*

Proof. Immediate consequence of the last two lemmas. □

Theorem 8. *If X and Y are compact, then $X \times Y$ is compact.*

Proof. By the last proposition, the projection π_Y is proper and hence $X \times Y = \pi_Y^{-1}(Y)$ is compact. □

The next theorem is the philosophical reason for the introduction of proper maps. Loosely speaking, a continuous map is proper iff it maps points near to infinity to points near to infinity. Compare and contrast the non-constant polynomial maps and the exponential maps.

We have a characterization of proper maps between locally compact hausdorff spaces in terms of their one-point compactifications.

Given a locally compact noncompact hausdorff space X, let $X_\infty := X \cup \{\infty\}$ where $\infty \notin X$. Let $\mathcal{T}$ denote the topology on X. Consider

$$\mathcal{T}_\infty := \mathcal{T} \cup \{V \subset X_\infty : X_\infty \setminus V \text{ is compact}\}.$$

Then
(i) $\mathcal{T}_\infty$ is a hausdorff topology on X_∞.
(ii) The subspace topology on X is $\mathcal{T}$.
(iii) $(X_\infty, \mathcal{T}_\infty)$ is compact.
(iv) X is dense in X_∞.

Theorem 9. *Let X and Y be locally compact hausdorff spaces. Then a continuous map $f : X \to Y$ is proper iff it extends to a continuous map of X_∞ to Y_∞ with $f(\infty_X) = \infty_Y$.*

Proof. Let f be proper. Extend f as above. Then we need to check its continuity. Let V be open in Y. The $f^{-1}(V)$ is an open subset of X and hence of X_∞. If $V \ni \infty_Y$, then $L := Y_\infty \setminus V$ is a compact subset of Y and hence $f^{-1}(L)$ is a compact subset of X, since f is proper. Since X is hausdorff, $f^{-1}(L)$ is closed. Hence $X \setminus f^{-1}(L)$ is open. But it is nothing but $f^{-1}(V)$.

Let f, the extension as in the statement, be continuous. Then $f^{-1}(Y) = X$, since $f(\infty_X) = \infty_Y$. If $L \subset Y$ is compact, then L is closed in Y and hence in Y_∞. So $f^{-1}(L)$ is closed in X_∞. Since X_∞ is compact, $f^{-1}(L)$ is compact. It is clearly a subset of X. Hence f is proper. $\square$

Proposition 10. *Let $f\colon X \to Y$ be a proper map (i) either between two locally compact hausdorff spaces or (ii) between two metric spaces. Then f is closed.*

Proof. Assume Case (i). Let g denote the extension of f to X_∞. If F is closed in Y, then $F_\infty := F \cup \{\infty_X\}$ is closed in X_∞ and hence is compact. Hence $g(F_\infty)$ is compact in Y_∞ and hence is closed, since Y_∞ is hausdorff. But then $f(F) = g(F_\infty) \cap Y$ is closed in Y. This proves the result in the first case.

We can also prove this directly without recourse to the one-point compactifcations as follows. Let C be closed in X. Let $q \in Y$ be a limit point of $f(C)$. Let V be an open set such that $q \in V$ and $L := \overline{V}$ is compact. (This is possible since Y is locally compact and hausdorff.) Consider $K := f^{-1}(L)$. Then K is closed, since f is proper. As $K \cap C$ is compact, we have $f(K \cap C) = L \cap f(C)$ (verify!) is compact and hence closed since Y is hausdorff. Since $q \in \overline{f(C)}$, and V is an open neighbourhood of q, we see that

$$q \in \overline{L \cap f(C)} = L \cap f(C) = f(K \cap C) \subset f(C).$$

This shows that any limit point q of $f(C)$ lies in $f(C)$ and hence $f(C)$ is closed.

Assume that X and Y are metric spaces. Let $C \subset X$ be closed. Let w be a limit point of $f(C)$. Then there exists a sequence $w_n \in f(C)$ such that $w_n \to w$. Since $w_n \in f(C)$, there exists $z_n \in C$ such that $w_n = f(z_n)$. Now the subset $L := \{w_n : n \in \mathbb{N}\} \cup \{w\}$ is a compact subset of Y. Since f is proper, its inverse image $K := f^{-1}(L)$ is compact. By our choice, (z_n) is a sequence in the compact set K and hence has a convergent subsequence, say, (z_{n_k}) converging to $z \in K$. Since C is closed, we conclude that $z \in C$. By continuity of f at z, we see that $f(z_{n_k}) \to f(z)$. Since $f(z_n) \to w$, it follows that $f(z) = w$. Hence we have shown that $w \in f(C)$, that is, $f(C)$ is closed. $\square$