Maps into the Punctured Plane

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The aim of this article is to classify the homotopy classes of maps from a circle to the punctured plane $\mathbb{C}^* := \mathbb{C} \setminus \{0\}$. Such a classification can be obtained from the knowledge of the fundamental group $\pi_1(S^1)$ of the circle. Our approach will be more analytic and will yield an alternative proof of the isomorphism $\pi_1(S^1) \simeq \mathbb{Z}$.

Definition 1. Let f and g be continuous functions from a space X to Y. Then f and g are *homotopic* iff there is a continuous function $H\colon I \times X \to Y$ such that $H(0, x) = f(x)$ and $H(1, x) = g(x)$ for all $x \in X$. H is called a homotopy from f to g. Thus a homotopy enables one to *pass continuously* from one map to another.

Let X be a topological space. We consider maps from X into $\mathbb{C}^*$. Such functions form a group under pointwise multiplication.

Definition 2. A map $f\colon X \to \mathbb{C}^*$ is an *exponential* if $f = \exp(g) = e^g$ for some continuous map $g\colon X \to \mathbb{C}$.

Ex. 3. The exponential maps form a subgroup of the group of maps from X to $\mathbb{C}^*$.

Theorem 4. *It is impossible to make a continuous choice $\theta(z) \in \arg(z)$ on $\mathbb{C}^*$. That is, there is no continuous map $\theta\colon \mathbb{C}^* \to \mathbb{R}$ such that $z = |z| \exp(\theta(z))$ for $z \in \mathbb{C}^*$.*

Proof. Assuming such a θ exists, consider $f\colon [0, 2\pi] \to \mathbb{R}$ by setting $f(t) := [\theta(e^{it}) + \theta(e^{-it})]/2\pi$. Then f is a real valued continuous function on $[0, 2\pi]$. Then $2\pi f(t)$ is a choice of $\arg(e^{it}e^{-it})$ and hence of $\arg(1)$. Thus it is integer valued continuous function on the interval $[0, 2\pi]$. By intermediate value theorem, it is a constant. In particular, $f(0) = f(\pi)$. This implies that $[\theta(1) + \theta(1)]/2\pi = [\theta(-1) + \theta(-1)]/2\pi$. Or, $\theta(1) = \theta(-1)$, which is impossible as the $\arg(1)$ and $\arg(-1)$ are disjoint. $\square$

However, the following lemma says that it is possible to assign the argument of a complex number in a continuous fashion if we restrict ourselves to $\mathbb{C}$ minus $\{z \in \mathbb{C} : \operatorname{Re} z \le 0\}$, or the complex plane minus any closed half line starting from the origin.

Lemma 5. *There exists a continuous map*

$$\alpha\colon X := C \setminus \{z \in \mathbb{C} : z \in \mathbb{R} \;\&\; z \le 0\} \to (-\pi, \pi)$$

such that $z = |z| e^{i\alpha(z)}$ for all $z \in X$.

Proof. Let us define the following open half-planes whose union is X: $H_1 := \{z \in \mathbb{C} : \operatorname{Re} z > 0\}$, $H_2 := \{z \in \mathbb{C} : \operatorname{Im} z > 0\}$ and $H_3 := \{z \in \mathbb{C} : \operatorname{Im} z < 0\}$. We define α_i on H_i which glue together to give the required map.

Let $z \in H_1$. Then $\operatorname{Re} z = |z| \cos\theta$ for some $\theta \in [-\pi, \pi]$ and hence $\cos\theta > 0$. This means that $\theta \in (-\pi/2, \pi/2)$. sin is increasing on $(-\pi/2, \pi/2)$ so that we have the continuous inverse $\sin^{-1}\colon (-1, 1) \to (-\pi/2, \pi/2)$. We define $\alpha_1(z) := \sin^{-1}(\frac{\operatorname{Im} z}{|z|})$. We can similarly define $\alpha_2\colon H_2 \to (0, \pi)$ and $\alpha_3\colon H_3 \to (-\pi, 0)$ by

$$\begin{aligned} \alpha_2(z) &= \cos^{-1}(\frac{\operatorname{Re} z}{|z|}) \\ \alpha_3(z) &= \cos^{-1}(\frac{\operatorname{Re} z}{|z|}). \end{aligned}$$

One easily sees that they agree upon their common domains. Thus we get the required function α. ◻

Every continuous function f from X to positive reals is an exponential. In this case $f = e^g$ where $g := \log f$. More generally we have

Lemma 6. *Suppose $f\colon X \to \mathbb{C}^*$ is a map that omits the negative real axis (that is, $f(X) \cap (-\infty, 0] = \emptyset$). Then f is an exponential.*

Proof. We use the previous lemma. Recall that the principal logarithm Log is defined on the given open subset of $\mathbb{C}$ by $\operatorname{Log} z = |z| e^{i\theta}$, where $\theta \in (-\pi, \pi)$. Thus Log depends continuously on z. If we set $g(z) := \operatorname{Log} f$, then we have $f = e^g$. ◻

The following result is related to Rouche's theorem in Complex Analysis.

Theorem 7. *Let f and g be functions from X to $\mathbb{C}$ satisfying*

$$|f(x) - g(x)| < |f(x)| + |g(x)|, \qquad x \in X. \tag{1}$$

Then f/g and g/f are exponential. In particular f is an exponential iff g is.

Proof. Observe that the strict inequality in Eq. 1 implies that neither f nor g can vanish on X. Dividing Eq. 1 by $f(x)$ we obtain

$$|1 - g(x)/f(x)| < 1 + |g(x)/f(x)|, \qquad x \in X.$$

It follows that g/f cannot assume negative real values, for, then the RHS will equal the LHS. Hence by Lemma 6, g/f is an exponential. As Eq. 1 is symmetric in f and g this means that f/g is also an exponential. The last statement is a consequence of the fact that the product of exponentials is an exponential. ◻

Theorem 8. *Let X be a compact metric space and $f, g\colon X \to \mathbb{C}^*$. Then f and g are homotopic iff f/g is an exponential.*

Proof. Suppose that f/g is an exponential, say, $f/g = e^h$. Then $F(t,x) := g(x)e^{th(x)}$ defines a homotopy from f to g.

Conversely, suppose that f and g are homotopic. Let F be a homotopy from f to g. Since $[0,1] \times X$ is compact, the continuous positive function $|F|$ attains its minimum. The minimum $m := \inf\{|F(t,x| : t \in [0,1], x \in X\}$ is positive. F is uniformly continuous on the compact metric space $I \times X$. Thus, for $\varepsilon := m$ there exists a $\delta > 0$ such that

$$|s-t| < \delta \Longrightarrow |F(s,x) - F(t,x)| < m, \qquad \forall x \in X. \tag{2}$$

We now choose an integer $N > 1/\delta$ and consider the maps $f_j \colon X \to \mathbb{C}^*$, defined by $f_j(x) := F(j/N, x)$. Now $f_0 = f$ and $f_N = g$. We see from Eq. 2 that

$$|f_j(x) - f_{j-1}(x)| < m \le |f_j(x)|, \qquad x \in X, 1 \le j \le N.$$

By Thm. 7, each f_j/f_{j-1} is an exponential. As $f/g = (f_0/f_1)(f_1/f_2)\cdots(f_{N-1}/f_N)$, we see that f/g is an exponential. □

Corollary 9. *Let X be a compact metric space and $f \colon X \to \mathbb{C}^*$. Then f is an exponential iff f is homotopic to a constant map.* □

Definition 10. A space is said to be *contractible* if there is a homotopy between the identity map and a constant map.

Ex. 11. Any convex subset of $\mathbb{R}^n$ is contractible.

Corollary 12. *Let X be a compact contractible metric space. Then every map f from X to $\mathbb{C}^*$ is an exponential.*

Proof. Let $F \colon [0,1] \times X \to X$ be the homotopy of the identity map of X and a constant map x_0. Then $f \circ F$ is a homotopy of f to the constant map $f(x_0)$. By Cor. 9, f is an exponential. □

Now we restrict our attention to maps of S^1 to $\mathbb{C}^*$. We wish to assign to any such map an index that corresponds to the number of times the functions wraps around the origin.

Definition 13. Let $f \colon S^1 \to \mathbb{C}^*$ be a map. Consider the map $\theta \mapsto f(e^{i\theta})$ of $[0, 2\pi]$ into $\mathbb{C}^*$. Since the interval is contractible, by Corollary 12,

$$f(e^{i\theta}) = e^{g(\theta)} \text{ for some } g \colon [0, 2\pi] \to \mathbb{C}^*. \tag{3}$$

Let $g_1 \colon [0, 2\pi] \to \mathbb{C}^*$ be another map which satisfies Eq. 3. Then $e^{g(\theta) - g_1(\theta)} = 1$. Hence $g(\theta) - g_1(\theta)$ must assume values from the discrete set $2\pi i\mathbb{Z}$. Since $g - g_1$ is continuous, it follows that $g - g_1$ is a constant. Thus the number $g(2\pi) - g(0)$ is independent of the choice of g satisfying Eq. 3. Consequently the number

$$\text{ind}\,(f) := [g(2\pi) - g(0)]/2\pi i$$

is well defined. This integer is called the index of the map f.

Ex. 14. Let $f_n(z) = z^n$ for $n \in \mathbb{Z}$. Then $\text{ind}\,(f) = n$.

Theorem 15. *The index function, defined on the maps from S^1 to $\mathbb{C}^*$ has the following properties:*

(i) $\operatorname{ind}(fg) = \operatorname{ind}(f) + \operatorname{ind}(g)$.

(ii) $\operatorname{ind}(f) = 0$ *iff f is an exponential.*

(iii) $\operatorname{ind}(f/|f|) = \operatorname{ind}(f)$.

(iv) If $f\colon S^1 \to S^1$ is a map such that $f(1) = 1$, then $\operatorname{ind}(f)$ *coincides with that of the loop α defined by $\alpha(s) = f(e^{2\pi is})$, $0 \leq s \leq 1$.*

Proof. (i) is easy and left to the reader.

Suppose $f(e^{i\theta}) = e^{h(e^{i\theta})}$. If we set $g(t) = h(e^{it})$ then g satisfies Eq. 3. Since $g(2\pi) = g(0)$, $\operatorname{ind}(f) = 0$. Conversely, assume that $\operatorname{ind}(f) = 0$. Write $f(e^{it}) = e^{ig(t)}$ for $0 \leq t \leq 2\pi$. Then $g(0) = g(2\pi)$ so that the function $h\colon S^1 \to \mathbb{C}$ defined by setting $h(e^{it}) = g(t)$, $0 \leq t \leq 2\pi$, is well defined and continuous. Since $f = e^h$ f is an exponential. This proves (ii).

Since $|f|$ is an exponential, $\operatorname{ind}(|f|) = 0$ by (ii). By (i), $\operatorname{ind}(f) = \operatorname{ind}(f/|f|) + \operatorname{ind}(|f|)$. (iii) follows.

Let f and α be as in (iv). Choose $h\colon [0,1] \to \mathbb{R}$ such that $h(0)$ and $\alpha(s) = e^{2\pi ih(s)}$, for $0 \leq s \leq 1$. Thus h is a lift of α and hence index $\operatorname{ind}(\alpha) = h(1)$. Define $g\colon [0, 2\pi] \to \mathbb{C}$ by $g(t) := 2\pi ih(t/2\pi)$. Then g satisfies Eq. 3 so that

$$\operatorname{ind}(f) = [g(2\pi) - g(0)]/2\pi i = h(1) = \operatorname{ind}(\alpha).$$

This proves (iv). □

Theorem 16. *Let $f, g\colon S^1 \to \mathbb{C}^*$ be maps. Then the following are equivalent:*

(i) f is homotopic to g.

(ii) $\operatorname{ind}(f) = \operatorname{ind}(g)$.

(iii) f/g is an exponential.

Proof. The equivalence of (i) and (iii) is a special case of Thm. 8.

If f/g is an exponential, then by (ii) of Thm. 15, $\operatorname{ind}(f/g) = 0$. Write $f = g\cdot(f/g)$. By (i) of Thm. 15, we obtain $\operatorname{ind}(f) = \operatorname{ind}(f/g) + \operatorname{ind}(g) = \operatorname{ind}(g)$. Conversely, if $\operatorname{ind}(f) = \operatorname{ind}(g)$, then $\operatorname{ind}(f/g) = 0$. So, by (ii) of Thm. 15, f/g is an exponential. □

Corollary 17. *Each map $f\colon S^1 \to \mathbb{C}^*$ is homotopic to precisely one of the maps $f_m\colon z \mapsto z^n$ where $n = \operatorname{ind}(f)$.* □

Corollary 18. *We have $\pi_1(S^1, 1) \equiv \mathbb{Z}$.*

Proof. Since the maps z^n are not homotopic, the loops $\alpha_n\colon [0,1] \to S^1$ defined by $\alpha_n(t) = e^{2\pi int}$ cannot be homotopic with end points fixed. On the other hand, let $\alpha\colon [0,1] \to S^1$ be an arbitrary loop based at 1. Define $f\colon S^1 \to S^1$ by $f(e^{2\pi is}) = \alpha(s)$. Let $n := \operatorname{ind}(f)$. By Thm. 16, f/α_n is an exponential, say, $f(e^{2\pi is})/e^{2\pi ins} = e^{h(e^{2\pi is})}$ for $0 \leq s \leq 1$. Then $F(t,s) := e^{th(e^{2\pi is})}e^{2\pi ins}$ for $0 \leq s, t \leq 1$ is a homotopy from α_n and the loop α with end points fixed. Thus the correspondence $\varphi\colon [\alpha_n] \mapsto n$ is a bijection between $\pi_1(S^1, 1)$ and $\mathbb{Z}$. One easily checks that the product path $\alpha_m\alpha_n$ corresponds to a map from S^1 to itself of index $m + n$, so that $\alpha_m\alpha_n$ is homotopic to a_{m+n}. Thus φ is a group homomorphism. □

Theorem 19 (Fundamental Theorem of Algebra)**.** *A polynomial* $p(z) := z^n + a_{n-1}z^{n-1} + \cdots + a_1 z + a_0$ *of degree* $n \geq 1$ *and with complex coefficients has a zero in* $\mathbb{C}$*.*

Proof. Choose R so large that

$$|\frac{a_{n-1}}{R}w^{n-1} + \cdots + \frac{a_1}{R^{n-1}} + \frac{a_0}{R^n}| < 1, \qquad |w| \leq 1.$$

This can be done, if, for instance, we take $R > |a_{n-1}| + \cdots + |a_0| + 1$. Define a map $g\colon B[0,1] \to \mathbb{C}$ by setting

$$g(w) := \frac{p(Rw)}{R^n} = w^n + \frac{a_{n-1}}{R}w^{n-1} + \cdots + \frac{a_0}{R^n}, \qquad |w| \leq 1.$$

The estimate above shows that $|g(w) - w^n| < 1$ for $w = 1$. By Thm. 7, the restriction of w^n/g to the unit circle is an exponential. Now w^n is not an exponential since its index is $n \geq 1$ (Theorem 15). Hence $g = w^n \cdot (g/w^n)$ is not an exponential. Corollary 12 shows that g must have a zero on $B[0,1]$, and hence p has a zero in $\mathbb{C}$. □

We now apply some of our earlier results to arrive at some standard theorems of the topology of the plane.

Corollary 20. *Assume that* $f\colon S^1 \to S^1$ *is homotopic to a constant map. Then there is a continuous function* $\varphi\colon S^1 \to \mathbb{R}$ *such that* $f(x) = e^{i\varphi(x)}$ *for all* $x \in S^1$*.*

Proof. A special case of Corollary 9. □

Theorem 21. *The circle* S^1 *is not contractible.*

Proof. If it were, then by Corollary 20 there is a function $\varphi\colon S^1 \to \mathbb{R}$ such that $Id(x) \equiv x = e^{i\varphi(x)}$ for all $x \in S^1$. Thus, there is a continuous argument on S^1 and hence on $\mathbb{C}^*$, contradicting Theorem 4.

Or, more directly, such a φ is 1-1 and in particular $\varphi(x) \neq \varphi(-x)$. Define $g\colon S^1 \to \{\pm 1\}$ by

$$g(x) := \frac{\varphi(x) - \varphi(-x)}{|\varphi(x) - \varphi(-x)|}.$$

Then g maps S^1 continuously *onto* $\{\pm 1\}$. This contradicts the connectedness of S^1. □

Definition 22. A subset A of a space X is a *retract* of X if there is a continuous function $r\colon X \to A$ such that $r(a) = a$ for all $a \in A$. r is called a retraction of X onto A.

Corollary 23. *There is no retraction of* $\mathbb{R}^2$ *onto* S^1*.*

Proof. Let $r\colon \mathbb{R}^2 \to S^1$ be retraction. Let $p = (0,0)$. Define a homotopy $H\colon I \times S^1 \to \mathbb{R}^2$ by $H(t,x) = tp + (1-t)x$. Then $r \circ H\colon I \times S^1 \to S^1$ is a contraction — contradicting Thm. 21. □

Corollary 24 (Brouwer Fixed Point Theorem)**.** *Let* $f\colon B[0,1] \to B[0,1]$ *be a continuous map. Then* f *has a fixed point, i.e., there is an* $x \in B[0,1]$ *such that* $f(x) = x$*.*

Proof. If there is no point x such that $f(x) = x$, then the two distinct points $f(x)$ and x determine a line joining $f(x)$ and x. We let $g(x)$ be the point on the boundary at which the line starting from $f(x)$ and going to x meets S^1. Then g is a retraction of $B[0,1]$ onto S^1—a contradiction to Corollary 23. In analytical terms, we have $g(x) = x + tv$, where $v = \frac{x - f(x)}{\|x - f(x)\|}$ and $t = -\langle x, v\rangle + \sqrt{1 - \|x\|^2 + (\langle x, v\rangle)^2}$. $\square$

We end this article with some exercises.

Ex. 25. Let X be compact and $f\colon X \to \mathbb{C}^*$ be an exponential. Show that there exists $\varepsilon > 0$ such that any map $g\colon X \to \mathbb{C}^*$ which satisfies $|f(x) - g(x)| < \varepsilon$ is an exponential.

Ex. 26. Let X be locally compact hausdorff space. Show that two maps f and g from X to $\mathbb{C}^*$ are homotopic iff f/g is an exponential. *Hint:* Consider first the case when X is compact.

Ex. 27. Let X be a locally compact contractible metric space. Show that any map $f\colon X \to \mathbb{C}^*$ is an exponential.

Ex. 28. Let $f\colon S^1 \to \mathbb{C}^*$ be given. Show that there exists a $\varepsilon > 0$ such that any map $g\colon S^1 \to \mathbb{C}^*$ with $|f(z) - g(z)| < \varepsilon$ for $z \in S^1$ has the same index as f.

Ex. 29. Assume that $f, g\colon S^1 \to S^1$ be maps such that f and g do not assume antipodal values at any point of S^1. Show that $\operatorname{ind}(f) = \operatorname{ind}(g)$.

Ex. 30. Show that any map from S^n $(n \geq 2)$ to $\mathbb{C}^*$ is an exponential.

Ex. 31. Show that any map from $\mathbb{P}^n(\mathbb{R})$ $(n \geq 2)$ to $\mathbb{C}^*$ is an exponential. (Note that $\mathbb{P}^n$ is not simply connected. Can you explain what is happening here?)

Ex. 32. Classify the maps from the figure eight to $\mathbb{C}^*$.