Examples of Cosets and Quotient Groups

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G acts transitively on $X\,(= \{aH/a \in G\})$. Find a G-set Y which is isomorphic to X.

Let $\{g_1B, g_2B, \ldots, g_kB\}$ be a partition of G, where B is a subset of G. Then, there exists i such that g_iB is a subgroup of G. It follows that this partition is a coset decomposition.

<u>Warm-up for cosets</u>

1. $G = \left(\mathbb{R}^2, +\right)$ $\quad H = \{y = 0\} = X$ axis
2. $G = \mathbb{C}^*$ $\quad H = S^1 = \{z \in \mathbb{C}/|z| = 1\}$
3. $G = \mathbb{Z}$ $\quad H = m\mathbb{Z}$
4. $H \leq G$ $\quad$ Define $\forall x, y \in G, \quad x \sim y \;$ if $\; xH = yH$.
5. $G = (\mathbb{R}^*, \cdot)$ $\quad H = \{\pm 1\}$

<u>Quotient groups</u>

1. $G = \left(\mathbb{R}^2, +\right)$ $\quad H = \{y = 0\} = X$ axis $\quad {}^{G}\!/_{H} \cong (\mathbb{R}, +)$ $\quad (x, y) \to y$
2. $G = \mathbb{C}^*$ $\quad H = S^1$ $\quad {}^{G}\!/_{H} \cong (\mathbb{R}^+, \cdot)$ $\quad z \to |z|$
3. $G = \mathbb{C}^*$ $\quad H = (\mathbb{R}^+, \cdot)$ $\quad {}^{G}\!/_{H} \cong S^1$ $\quad z \to \frac{z}{|z|}$
4. $G = \mathbb{Z}$ $\quad H = n\mathbb{Z}$ $\quad \zeta = e^{\frac{2\pi i}{n}}$ $\quad {}^{G}\!/_{H} \cong <\zeta>$ $\quad \bar{a} \to \zeta^a$
5. $G = GL\,(n, \mathbb{R})$ $\quad H = SL\,(n, \mathbb{R})$ $\quad {}^{G}\!/_{H} \cong (\mathbb{R}^*, \cdot)$ $\quad A \to det(A)$
6. $G = (\mathbb{R}^*, \cdot)$ $\quad H = \{\pm 1\}$ $\quad {}^{G}\!/_{H} \cong (\mathbb{R}^+, \cdot)$ $\quad x \to |x|$ $\quad x \to x^2$
7. $G = (\mathbb{C}^*, \cdot)$ $\quad H_n = \{n^{\text{th}}$ roots of unity$\}$ $\quad {}^{G}\!/_{H_n} \cong G$ $\quad z \to z^n$

8. $G = (\mathbb{R}, +)$ $H = 2\pi\mathbb{Z}$ ${}^{G}\!/_{H} \cong S^1$ $x \to e^{ix}$
9. $G = (\mathbb{R}, +)$ $H_1 = \mathbb{Z}$ ${}^{G}\!/_{H_1} \cong S^1$ $x \to e^{i2\pi x}$

   Caution: $H \cong H_1 \not\Rightarrow {}^{G}\!/_{H} \cong {}^{G}\!/_{H_1}$

   $m, n \in \mathbb{Z}\backslash\{0\} \Rightarrow m\mathbb{Z} \cong n\mathbb{Z}$. But
   ${}^{\mathbb{Z}}\!/_{m\mathbb{Z}} \cong {}^{\mathbb{Z}}\!/_{n\mathbb{Z}} \Leftrightarrow m = \pm n$
10. $G = (\mathbb{C}^*, \cdot)$ $H = (\mathbb{R}^*, \cdot)$ ${}^{G}\!/_{H} \cong S^1$ $z \to (\frac{z}{|z|})^2$
   From (7), $H_n \ncong H_m$ for $n \neq m$ but ${}^{G}\!/_{H_n} \cong {}^{G}\!/_{H_m}$
   Thus, ${}^{G}\!/_{H_1} \cong {}^{G}\!/_{H_2} \not\Rightarrow H_1 \cong H_2$.

In most of the examples, ${}^{G}\!/_{H}$ is isomorphic to a subgroup of G.

However, ${}^{\mathbb{Z}}\!/_{n\mathbb{Z}}$ for $n \neq 0, \pm 1, \pm 2$ gives an example to the contrary.

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