Cantor Set

S. Kumaresan
School of Math. and Stat.
University of Hyderabad
Hyderabad 500046
kumaresa@gmail.com

The geometric ideas behind the decimal, binary and ternary expansions of any real number depend on the nested interval theorem. We recall it in a form useful to us.

Theorem 1 (Nested Interval Theorem). *Let $I_k \subset \mathbb{R}$ be finite intervals with endpoints a_k and b_k such that (i) $I_{k+1} \subset I_k$ for each $k \in \mathbb{N}$ and (ii) $\lim_{k\to\infty} \ell(I_k) = \lim_{k\to\infty} b_k - a_k \to 0$. Then there exists a unique $c \in \mathbb{R}$ such that $a_k \leq c \leq b_k$ for all k.*

Proof. Consider the set $A := \{a_k : k \in \mathbb{N}\}$. This set is nonempty, bounded above by each of b_n. Hence by the least upper bound property of $\mathbb{R}$, there exists $c \in \mathbb{R}$ such that $c = \sup A$. Then $c \leq b_n$, since c is the l.u.b. of A and each b_n is an upper bound for A. Also, since c is an upper bound for A, $a_n \leq c$ for all n. Thus, $a_n \leq c \leq b_n$.

If d is also such that $a_n \leq d \leq b_n$ for each n, then, $c, d \in [a_n, b_n]$ for each n. From this we conclude that $|c - d| \leq b_n - a_n$ for all n. As $b_n - a_n \to 0$, it follows that $|c - d| = 0$. Hence $c = d$. □

Remark 2. Let us reiterate that we did not assume that the intervals I_k are closed. However, our conclusion was that $c \in [a_n, b_n]$ for all n. We do **not** claim that $c \in I_n$. An easy example is $I_k = (0, 1/k)$.

How do we plan to use this? Let $p \in \mathbb{N}$ be greater than or equal to 2. Then we want to show that there exists p-expansion for any real number x in the following sense: There exists an integer x_0 and numbers a_k lying in $\{0, 1, \dots, p-1\}$ such that

$$x = x_0 + \sum_k \frac{a_k}{p^k}.$$

We may assume without loss of generality that $x \geq 0$. For, $x < 0$ and if $y = -x = y_0 + \sum_k \frac{a_k}{p^k}$ is the p-expansion of y, then $x = -(y_0 + 1) + \sum_{k=1}^{\infty} \frac{a_k}{p^k}$ is the p-expansion for x. Any real number x can be written in the form $x = x_0 + a$, where $x_0 \in \mathbb{Z}$ and $a \in [0, 1)$. This suggests that it suffices to consider only $x \in [0, 1)$.

Let $x \in [0, 1)$ be given. The key geometric idea is to subdivide $[0, 1)$ into p-equal parts, choose the one which contains our element x, then subdivide this subinterval into p-equal parts choose the one in which x lies and so on. We are mostly interested when $p = 2, 3, 10$.

When $p = 2, 3, 10$, the expansions are respectively called *binary*, *ternary* and *decimal*. To keep our notation simple, let us concentrate on $p = 3$ and describe the ternary expansion.

Let $x \in [0, 1)$ be given. We can subdivide $[0, 1)$ into three disjoint subintervals: $[0, 1) = [0, 1/3) \cup [1/3, 2/3) \cup [2/3, 1)$. . Now x lies in exactly one of the subintervals, $I_0 := [0, 1/3)$, $I_1 := [1/3, 2/3)$ and $I_2 := [2/3, 1)$. If it lies in I_k, we let $a_1 = k$. For example, if we consider, $x = 10/27$, then $a_1 = 1$ whereas if $x = 2/3$, then $a_1 = 2$. We then subdivide I_k into three equal parts, as earlier: if $I_k = [c, d)$, (where $d = c + \frac{1}{3}$ then the subintervals are $[c/3, (c/3) + 1/9)$, $[(3c + 1)/9, (3c + 2)/9)$ and $[(3c + 2)/9, d)$. Then x lies in precisely one of them. We let a_2 to be the right endpoint of this subinterval. We carry out this process ad inf. Thus we get a sequence of intervals

$$J_n := [\frac{a_1}{3} + \cdots + \frac{a_{n-1}}{3^{n-1}} + \frac{a_n}{3^n}, \frac{a_1}{3} + \cdots + \frac{a_{n-1}}{3^{n-1}} + \frac{a_n + 1}{3^n}),$$

such that $x \in J_n$. Note that $0 \leq k \leq 2$. By nested interval theorem, $x \in J_n$ for each n. Hence $x = \sum_n \frac{a_n}{3^n}$. (Exercise: Prove this.)

Another subdivision of $(0, 1]$ (note that it is not closed on the left but on the right) which is also used is: $(0, 1] = (0, 1/3] \cup (1/3, 2/3] \cup (2/3, 1]$. The subinterval in which x lies is further subdivided into three equal parts as per the new recipe. In this process, the first digit in the ternary expansion of $x = 1/3$ is 0, and all other digits are 2. Thus, we have $1/3 = 0.1(\text{ternary}) = 0.0222\cdots(\text{ternary})$.

Thus in both the processes, the digits are defined by locating the point x in a sequence of intervals, the length of which go down by a factor of three each time:

$$\begin{aligned}
a_1 &= \sup\left\{k \in \mathbb{Z} \;\middle|\; \frac{k}{3} \leq x\right\} \\
b_1 &= \sup\left\{k \in \mathbb{Z} \;\middle|\; \frac{k}{3} < x\right\} \\
a_2 &= \sup\left\{k \in \mathbb{Z} \;\middle|\; \frac{a_1}{3} + \frac{k}{3^2} \leq x\right\} \\
b_2 &= \sup\left\{k \in \mathbb{Z} \;\middle|\; \frac{a_1}{3} + \frac{k}{3^2} < x\right\} \\
\vdots \quad & \quad \vdots \\
a_n &= \sup\left\{k \in \mathbb{Z} \;\middle|\; \frac{a_1}{3} + \cdots + \frac{a_{n-1}}{3^{n-1}} + \frac{k}{3^n} \leq x\right\} \\
b_n &= \sup\left\{k \in \mathbb{Z} \;\middle|\; \frac{a_1}{3} + \cdots + \frac{a_{n-1}}{3^{n-1}} + \frac{k}{3^n} < x\right\}
\end{aligned}$$

Note that $a_n, b_n \in \{0, 1, 2\}$. We then have $x = 0.a_1 a_2 \cdot a_n \cdots = 0.b_1 b_2 \cdots b_n \cdots$. We refer to the first (resp. the second) form as the terminating (resp. nonterminating) ternary expansion of x. (As in the case of decimals, one can characterize those x that admit both expansions. We leave the investigation to the reader.) For instance, if $x = \frac{10}{27}$, then $x = 0.101 = 0.100222\ldots$ (ternary). Note that $a_k \in \{0, 1, 2\}$.

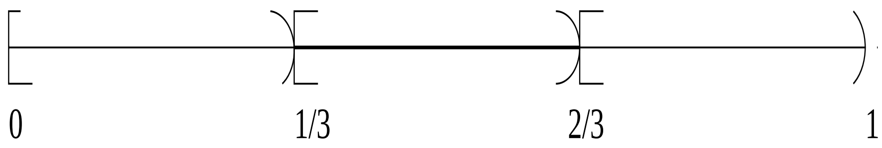
With this brief introduction, we are ready to define Cantor ternary set. We shall give analytical definition first and explain the geometric construction later.

Consider the interval $[0,1]$, represented in the nonterminating ternary form: for each $x \in [0,1]$, $x = \sum_{k=1}^{\infty} \frac{a_k}{3^k}$, where $a_k \in \{0,1,2\}$ for all $k \in \mathbb{N}$. The Cantor (ternary) set K consists of all those $x \in [0,1]$ which have a ternary representation $\sum_{k=1}^{\infty} \frac{a_k}{3^k}$, where $a_k \in \{0,2\}$ for all $k \in \mathbb{N}$.

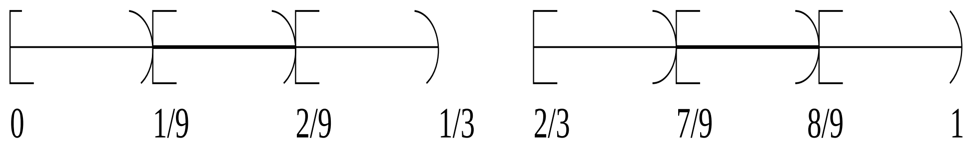
The geometric construction behind K is as follows:

First a piece of notation. Given an interval $[a,b]$, the interval $(a + \frac{1}{3}(b-a), b - \frac{1}{3}(b-a))$ is called the middle third open interval of $[a,b]$.

Take $[0,1]$ and delete the open middle third interval $J_{11} := (\frac{1}{3}, \frac{2}{3})$. This deletes numbers with 1 in the first ternary place. (The numbers $\frac{1}{3}$ and 1 are not deleted as they have ternary expansions $0.0222\ldots$ and $0.222\ldots$ respectively).



Step 1:



step 2:

Take the two remaining closed intervals $[0, \frac{1}{3}]$ and $[\frac{2}{3}, 1]$ and delete the open middle thirds of these intervals. They are $J_{21} := (1/9, 2/9)$ and $J_{22} := (7/9, 8/9)$. In this step we have deleted numbers with 1 in the second ternary place. (The numbers $\frac{1}{9}$ and $\frac{7}{9}$ are not deleted, because they have ternary expansions $0.00\overline{2}$ and $0.20\overline{2}$ respectively.)

Continuing this way, after n steps, we would have deleted $1 + 2 + \cdots + 2^{n-1} = 2^n - 1$ disjoint open intervals J_{kr} where $1 \le k \le n$ and $1 \le j \le 2^{k-1}$. We are left with 2^n closed intervals each of length 3^{-n}.

We continue this process ad infinitum. What remains is the Cantor set K:

$$K := [0,1] \setminus \cup_{k=1}^{\infty} \cup_{r=1}^{2^k-1} J_{kr}.$$

K is closed: For, K was obtained by removing an open set $U := \cup_{k=1}^{\infty} \cup_{r=1}^{2^k-1} J_{kr}$ a union of open intervals.

We also note that K is contained in $[0,1] \setminus \cup_{k=1}^{n} \cup_{r=1}^{2^k-1}$ for each n. The latter set is a union of 2^n intervals sum of whose lengths is $(2/3)^n$. Thus K has "no length". However K has uncountably many points, more than the obvious points, viz., the end points of the deleted intervals.

This follows from the ternary expansion and the fact that the set of functions from $\mathbb{N}$ to $\{0, 2\}$ has the cardinality of $\mathbb{R}$. More specifically, consider the map $f : K \to [0, 1]$ given by

$$f(\sum_{k=1}^{\infty} \frac{a_k}{3^k}) = \sum_{k=1}^{\infty} \frac{a_k}{2} 2^{-k}.$$

Then f maps K onto $[0, 1]$. However f is not one-to-one. f is called the *Cantor-Lebesgue function*.

Remark 3. Note that at the n^{th} stage of the construction we have a closed set F_n which is the union of 2^n closed intervals of the form $[\frac{k}{3^n}, \frac{k+1}{3^n}]$ for specific k's. For example

$$\begin{aligned} F_1 &= [0, \frac{1}{3}] \cup [\frac{2}{3}, 1] \\ F_2 &= [0, \frac{1}{9}] \cup [\frac{2}{9}, \frac{1}{9}] \cup [\frac{6}{9}, \frac{7}{9}] \cup [\frac{8}{9}, 1] \end{aligned}$$

Thus K is $\cap F_n$. From this it follows that K contains no non-empty open interval. For, if $(a, b) \subset F_n$ for each n, then $|a - b| \leq \frac{1}{3^n}$ for all n and hence $a = b$ and hence $(a, b) = \emptyset$.

In particular, no connected subset of K can have more than one point.

Let $x \in K$. Then $x \in F_k$ for each k and is therefore a cluster point of the end points of the intervals in F_k. Thus K is a *perfect* set — a set each of whose points is its cluster point.