Quotient Topology

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This article is devoted to the mathematical formulation of gluing geometric objects to get new geometric objects. For example, one may form a circle from a closed line segment by bending it around and gluing the ends together. Or, one can form a cylinder from a rectangle by bending the rectangle around and gluing two opposite sides together. If we further bend the cylinder around and glue the two circular rims together we get a torus or a cycle tube. In this article, we concentrate on some of the very basic results of the theory which will enable the reader to deal with quotient spaces with confidence. The theory is full of pathologies and often text-books and teachers tend to frighten the beginner with the macabre rather than emphasizing the positive aspects and initiating him into a working knowledge of quotient spaces. This article attempts to make it easy for a student to learn quotient spaces.

Let X be a set and $\sim$ be an equivalence relation on X. Let $X/\sim$ be the quotient set or the set of equivalence classes of $\sim$. Let $\pi\colon X \to X/\sim$ be the *quotient map* defined by $\pi(x) = [x]$, the equivalence class of x. If we further assume that X is a topological space, we then want to introduce a topology on the quotient set so that the quotient map π is continuous. Note that the indiscrete topology on $X/\sim$ will be one such. However we would like to have the largest possible topology on $X/\sim$ with this property. If τ is such a topology and V is open in $X/\sim$, then $\pi^{-1}(V)$ must be open in X. This suggests the following

Definition 1. With the notation as above, we define τ to be the set of $V \subset X/\sim$ such that $\pi^{-1}(V)$ is open in X. It is easy to check that τ is indeed a topology on the quotient set. The space $(X/\sim, \tau)$ is called the quotient space of X relative to the equivalence $\sim$.

We record the following fact which is an immediate consequence of the definition of the quotient topology.

Proposition 2. *Let X be a topological space and $\sim$ an equivalence relation on X. Then the quotient topology on $X/\sim$ is the largest topology for which the natural quotient map $\pi\colon X \to X/\sim$ is continuous.* □

The next theorem, though easy, is quite often used to check the continuity of maps from quotient spaces to others.

Theorem 3 (Universal Mapping Property)**.** *Let $\pi\colon X \to X/\sim$ be a quotient map. A map $f\colon X/\sim \to Y$ is continuous iff $f \circ \pi$ is continuous.*

Proof. If f is continuous, then certainly so is $f \circ \pi$. To prove the converse, let V be an open set of Y. Then $(f \circ \pi)^{-1}(V) = \pi^{-1}(f^{-1}(V))$ is an open subset of X. By the definition of quotient topology, $f^{-1}(V)$ is open set in $X/\sim$. Hence f is continuous. $\square$

The next theorem tells us how to generate quotient spaces.

Theorem 4. *Let $f\colon X \to Y$ be continuous. Let $\sim$ be the equivalence relation on X defined by $x_1 \sim x_2$ iff $f(x_1) = f(x_2)$. Then there exists a continuous function $g\colon X/\sim \to Y$ such that $f = g \circ \pi$.*

Proof. Define $g([x]) = f(x)$ for any $x \in [x] \in X/\sim$. Then g is well-defined and $g \circ \pi = f$. Thm. 3 assures the continuity of g. $\square$

The following result allows us to identify quotient spaces with other concrete spaces.

Theorem 5. *Let X and Y be compact. Assume further that Y is hausdorff. Let $f\colon X \to Y$ be a surjective continuous map. Define the equivalence relation $\sim$ by declaring $x_1 \sim x_2$ iff $f(x_1) = f(x_2)$. Then $X/\sim$ is homeomorphic to Y.*

Proof. $X/\sim$ is compact, being the image of X under the continuous map π. Let $g\colon X/\sim \to Y$ be the continuous function defined by $g([x]) = f(x)$. Then g is continuous, 1-1 and onto. Hence by Exer. 6 g is a homeomorphism. $\square$

Ex. 6. Let $f\colon X \to Y$ be a 1-1 continuous map from a compact space to a Hausdorff space. Then f is a homeomorphism of X onto $f(X)$.

We give four simple applications to illustrate the power of Thm. 5.

Example 7. Consider the quotient space obtained from $[0, 1]$ got by identifying the end points 0 and 1. That is, Y is the space $X/\sim$ where $X := [0, 1]$ and the equivalence classes are $\{t\}$ for $0 < t < 1$ and $\{0, 1\}$. Consider the map $f\colon [0, 1] \to S^1$ given by $f(t) := e^{2\pi i t}$. Y and S^1 are homeomorphic by Thm. 5.

Example 8. We show that the quotient space got by identifying two of the opposite sides of a rectangle is homeomorphic to a cylinder. Let $X := \{(u, v) \in \mathbb{R}^2 : -\pi \le u \le \pi, -1 \le v \le 1\}$. Let $Y := \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 = 1, |z| \le 1\}$. On X we define the equivalence relation by setting $(u, v) \sim (u', v')$ iff $u = \pm\pi = u'$ and $v = v'$ if $u = \pm\pi = u'$ and otherwise $u = u'$ and $v = v'$. Consider the map $f\colon X \to Y$ given by $f(u, v) := (\cos u, \sin u, v)$. Then the level sets of f are precisely the equivalence classes. f induces a homeomorphism via Thm. 5.

Example 9. Let $X := [0, 1] \times [0, 1]$ and $f\colon X \to S^1 \times S^1$ be defined by $f(s, t) := (e^{2\pi i s}, e^{2\pi i t})$. The level sets of f are the singletons inside the square, the pairs of opposite points on the interiors of the bounding intervals and the set of four vertices of X. Thus we see that the quotient space of a square obtained from the equivalence is a torus.

For any space X and a subset A of X, the space X/A stands for the quotient space of X with respect to the equivalence: $x_1 \sim x_2$ iff $x_1 = x_2$ or $x_1, x_2 \in A$. Thus X/A is the space obtained from X by collapsing A to a single point.

Example 10. Consider the map $f\colon B[0,1] \subset \mathbb{R}^n \to S^n \subset \mathbb{R}^{n+1}$ defined by setting

$$f(tx) := (\cos \pi(1-t), x_1 \sin \pi(1-t), \ldots, x_n \sin \pi(1-t)),$$

where $x \in S^{n-1}$ and $0 \le t \le 1$. Then $f(S^{n-1}) = e_0 = (1, 0, \ldots, 0) \in S^n$. f induces a homeomorphism of $B[0,1]/S^{n-1}$ with S^n. See also Exer. 12 below.

Ex. 11. Let F be a closed subset of a compact Hausdorff space X. Prove that the quotient space obtained from X by identifying F to a single point is homeomorphic to the one-point compactification of $X \setminus F$.

Ex. 12. Let B be the closed unit ball in $\mathbb{R}^n$. Prove that the quotient space obtained from B by identifying its boundary S^{n-1} to a point is homeomorphic to the n-sphere.

Ex. 13. Let X denote the union of circles (in $\mathbb{R}^2$) centred at $(0, 1/n)$ and of radius $1/n$ with the subspace topology of $\mathbb{R}^2$. Let Y denote the quotient space $\mathbb{R}/\mathbb{Z}$ obtained from $\mathbb{R}$ by collapsing all of $\mathbb{Z}$ to a single point. Show that X and Y are not homeomorphic.

Thm. 5 is a special case of the next theorem which can be used the same way as the former was used in the examples above. Thm. 3, Thm. 5 and Thm. 14 are thus the most useful results in practice.

Theorem 14. *Let $f\colon X \to Y$ be an open (or closed) continuous surjective map. Then Y is homeomorphic to the quotient space of X obtained by identifying each level set of f to a point.*

Proof. Argue as in Thm. 5. □

The following exercises introduce some of the important quotient spaces. They will help the reader understand the concept of quotient spaces well.

Ex. 15 (Real Projective Spaces)**.** Let $\mathbb{P}^n(\mathbb{R})$ be the n-dimensional projective space over $\mathbb{R}$. It is the quotient space obtained from S^n with respect to the equivalence relation $x \sim y$ iff $x = \pm y$. Prove the following:
(a) $\mathbb{P}^n(\mathbb{R})$ is a compact Hausdorff space.
(b) The projection $\pi\colon S^n \to \mathbb{P}^n(\mathbb{R})$ is a local homeomorphism, that is, each $x \in S^n$ has an open neighbourhood which is mapped homeomorphically onto an open neighbourhood of $\pi(x)$ by π.
(c) $\mathbb{P}^1$ is homeomorphic to S^1.
(d) $\mathbb{P}^n(\mathbb{R})$ is homeomorphic to the quotient space obtained from the closed unit ball B in $\mathbb{R}^n$ by identifying the antipodal points of its boundary S^{n-1}.
(e) On $X := \mathbb{R}^{n+1} \setminus \{0\}$ introduce the equivalence relation $x \sim y$ iff there is a nonzero $\alpha \in \mathbb{R}$ such that $x = \alpha y$. Show that $X/{\sim}$ is homeomorphic to $\mathbb{P}^n(\mathbb{R})$. (Thm. 14 may be of use here.)

Ex. 16 (Complex Projective Spaces)**.** Think of $X := S^{2n+1}$ as a subset of $\mathbb{C}^{n+1}$:

$$S^{2n+1} = \{z \in \mathbb{C}^{n+1} : \sum_i |z_i|^2 = 1\}.$$

Define an equivalence relation on X by declaring that $z \sim w$ iff there exists a $\lambda \in S^1 \subset \mathbb{C}$ such that $z = \lambda w$. The resulting quotient space, denoted by $\mathbb{P}^n(\mathbb{C})$ is called the n-dimensional

complex projective space. Prove the following:
(a) $\mathbb{P}^n(\mathbb{C})$ is a compact Hausdorff space.
(b) $\mathbb{P}^1(\mathbb{C})$ is homeomorphic to S^2.
(c) Let π denote the projection $\pi\colon X \to \mathbb{P}^n(\mathbb{C})$. Show that $\pi^{-1}(x)$ is homeomorphic to S^1 for all $x \in \mathbb{P}^n(\mathbb{C})$. Show that each $x \in \mathbb{P}^n(\mathbb{C})$ has a neighbourhood U such that $\pi^{-1}(U)$ is homeomorphic to $U \times S^1$.
(d) Let $Y := \mathbb{C}^{n+1} \setminus \{0\}$. Let $\sim$ be the equivalence relation defined by $x \sim y$ iff there exists a nonzero $\lambda \in \mathbb{C}$ such that $x = \lambda y$. Then the quotient space $Y/{\sim}$ is homeomorphic to $\mathbb{P}^n(\mathbb{C})$. Thus $\mathbb{P}^n(\mathbb{C})$ can be regarded as the set of one-dimensional subspaces of $\mathbb{C}^{n+1}$.

Quotient spaces are full of pathologies. It is necessary to recognize which of the topological properties of X are inherited by $X/{\sim}$ and which are not. The following exercises deal with this concern.

Ex. 17. Let X be a topological space and $X/{\sim}$ be its quotient with respect to an equivalence relation. Prove the following:
(a) If X is compact, so is $X/{\sim}$.
(b) If X is connected, so is $X/{\sim}$.
(c) If X is path connected, so is $X/{\sim}$.

Ex. 18. Pathologies
(a) Let $\sim$ be an equivalence relation on a space X. Prove that $X/{\sim}$ is a T_1-space iff each equivalence class is closed. Give an example of a T_1 space X and a quotient space of X which is not T_1.
(b) Define an equivalence relation on $X := [0,1] \times [0,1]$ by setting $(s,t) \sim (s',t')$ iff $t = t' > 0$. Describe the quotient space and show that it is not Hausdorff.

One of the most important ways of defining equivalence relations is by means of group actions. So it should not be a surprise that many quotient spaces arise as the quotients of group actions on spaces. Below we indicate some of these instances.

Definition 19. We say a group G acts on a space X if there is a map $\varphi\colon G \times X \to X$ with the following properties (Below we write $g \cdot x$ for $\varphi(g,x)$):
(i) $e \cdot x = x$ for the identity $e \in G$ and $x \in X$.
(ii) For $g, h \in G$ and $x \in X$, we have $(gh) \cdot x = g \cdot (h \cdot x)$.

Note that these conditions mean that for each $g \in G$ the map $\varphi_g : x \mapsto g \cdot x$ is a homeomorphism of X onto itself. Thus we have a group homomorphism of G into the group of homeomorphisms of X.

We say X is a G-space if an action of G on X is given.

On any G-space, we have a natural equivalence: $x \sim y$ iff there exists a $g \in G$ such that $y = g \cdot x$. The equivalence classes are called the *orbits* of G, for, $[x] \equiv \{g \cdot x : g \in G\}$. The corresponding quotient space $X/{\sim}$ is denoted by X/G.

Ex. 20. Let $X = \mathbb{R}$ and $G = \mathbb{Z}$. Let G act on X by $n \cdot x = x + n$. Then the quotient space $\mathbb{R}/\mathbb{Z}$ is homeomorphic to S^1.

Ex. 21. Let $X = \mathbb{R}^2$ and $G = \mathbb{Z}$. G acts on X by $n \cdot (x,y) = (x+n, y)$. Show that the resulting quotient space is the infinite cylinder $\{(x,y,z) \in \mathbb{R}^3 : x^2 + y^2 = 1\}$.

Ex. 22. Let $X = S^n$ and $G = \mathbb{Z}_2$, the multiplicative group of two elements. If -1 is the nontrivial element of G, then define $-1 \cdot x := -x$. Then X/G is $\mathbb{P}^n(\mathbb{R})$.

In a similar way, if we let $G := \mathbb{R}^*$, the multiplicative group of nonzero reals act on $X := \mathbb{R}^{n+1} \setminus \{0\}$ via scalar multiplication, then X/G is the n-dimensional real projective space.

Ex. 23. Let $X := S^{2n+1} \subset \mathbb{C}^{n+1}$. Let $S^1 \subset \mathbb{C}$ act on X by

$$e^{it} \cdot (z_1, \dots, z_{n+1}) := (e^{it} z_1, \dots, e^{it} z_{n+1}).$$

The quotient S^{2n+1}/S^1 is homeomorphic to $\mathbb{P}^n(\mathbb{C})$.

Let $Y := \mathbb{C}^{n+1} \setminus \{0\}$. Let $G := \mathbb{C}^*$, the multiplicative group of complex numbers act on Y via scalar multiplication. The quotient Y/G is $\mathbb{P}^n(\mathbb{C})$.

Ex. 24. Let $X = \mathbb{R}^2$ and $G = \mathbb{Z}^2$. The action is $(m, n) \cdot (x, y) = (x + m, y + n)$. The quotient $\mathbb{R}^2/\mathbb{Z}^2$ is homeomorphic to the torus in $\mathbb{R}^3$ got by revolving around the z-axis a circle of unit radius centered at $(2, 0, 0)$ and of radius 1 in the (x, z)-plane. *Hint*: Consider the map $(u, v) \mapsto ((2 + \cos 2\pi u) \cos 2\pi v, (2 + \cos 2\pi u) \sin 2\pi v, \sin 2\pi u)$.

Ex. 25 (Möbius Strip)**.** On the unit square X we define the equivalence relation as follows:

$$(x, y) \sim (x', y') \iff (x, y) = (x', y') \text{ or } \{x, x'\} = \{0, 1\} \text{ and } y = 1 - y'.$$

Thus two points of opposite vertical sides are identified *cross-wise*. The quotient space is known as the Möbius strip.

Let $Y := \{(x, y) \in \mathbb{R}^2 : -1/2 \leq y \leq 1/2\}$. Let $\mathbb{Z}$ act on Y by $m \cdot (x, y) := (m + x, (-1)^m y)$. Show that the quotient space $Y/\mathbb{Z}$ is homeomorphic to the Möbius strip.

Ex. 26 (Klein Bottle)**.** Let X be the unit square. Define an equivalence relation on X whose nontrivial relations are given by

$$(0, y) \sim (1, y) \quad \text{and} \quad (x, 0) \sim (1 - x, 1).$$

The quotient space is called the *Klein's bottle*.

Let $Y = \mathbb{R}^2$. Let $\varphi, \psi \colon \mathbb{R}^2 \to \mathbb{R}^2$ be given by

$$\varphi(x, y) := (x + 1, y), \qquad \psi(x, y) := (1 - x, y + 1).$$

Thus φ is a translation parallel to the x-axis and ψ is a glide reflection along the line $x = 1/2$. Show that φ and ψ are homeomorphisms of $\mathbb{R}^2$, $\psi \circ \varphi = \varphi^{-1} \circ \psi$ so that $G := \{\varphi^m \psi^{2n} \psi^\epsilon : m, n \in \mathbb{Z}, \epsilon \in \{0, 1\}\}$ is a group of homeomorphisms of $\mathbb{R}^2$. Show that Y/G is homeomorphic to the Klein bottle.

Ex. 27. Can you identify the quotient spaces X/G?
(a) Let $S^1 \subset \mathbb{C}$ act on S^2 via rotations about the z-axis.
(b) Let $\mathbb{Z}_n$ act on S^2 via rotations by an angle which is a multiple of $2\pi/n$.
(c) Let $O(n)$, the orthogonal group, act on $\mathbb{R}^n$ via the usual linear action.

Ex. 28 (Lens Spaces)**.** Consider $S^3 \subset \mathbb{C}^2$. Let p and q be relatively prime integers. Let a generator $g \in \mathbb{Z}_p$ act on S^3 by $g \cdot (z_1, z_2) := (e^{2\pi i/p} z_1, e^{2\pi q i/p} z_2)$. The quotient space is denoted by $L(p,q)$ and called a Lens space. Show that $L(2,1)$ is homeomorphic to $\mathbb{P}^3(\mathbb{R})$. If p divides $q - q'$, then $L(p,q)$ is homeomorphic to $L(p,q')$.

If $L(p,q)$ and $L(p',q')$ are homeomorphic then $p = p'$. *Hint:* What is the fundamental group of X/G if X is simply connected and G acts properly discontinuosly? (G acts *properly discontinuously* on X if for any $x \in X$ there exists a neighbourhood U of x such that $g \cdot U \cap U = \emptyset$ for every $g \neq e$ in G.)

Ex. 29. Let $X := \mathbb{R}^n \setminus \{0\}$. Fix any real number $\alpha \notin \{0, \pm 1\}$. Let $G := \mathbb{Z}$ act on X by $m \cdot x := \alpha^m x$. Show that G acts properly discontinuously. Identify the quotient X/G.

Hausdorff Quotient Spaces

Definition 30. Let $\sim$ be an open equivalence on a space X. For any set $A \subset X$, we let $[A]$ stand for the set of all elements $x \in X$ which are equivalent to some element of A. The equivalence is called *open* if $[A]$ is open whenever A is open in X.

Ex. 31. An equivalence relation $\sim$ on a topological space is open iff the quotient map $\pi \colon X \to X/{\sim}$ is open. *Hint:* Observe that $[A] = \pi^{-1}(\pi(A))$.

Proposition 32. *Let $\sim$ be an equivalence on a topological space X. Then $R := \{(x,y) : x \sim y\}$ is closed in $X \times X$ iff the quotient space $X/{\sim}$ is hausdorff.*

Proof. Assume that $X/{\sim}$ is hausdorff and that $(x,y) \notin R$. Then there exist disjoint neighbourhoods U of $\pi(x)$ and V of $\pi(y)$. We denote by $\tilde{U}$ and $\tilde{V}$ the open sets $\pi^{-1}(U)$ and $\pi^{-1}(V)$, which contain x and y respectively. If the open set $\tilde{U} \times \tilde{V}$ containing (x,y) intersects R, then it must contain a point (x', y') for which $x' \sim y'$, so that $\pi(x') = \pi(y')$, contrary to the assumption that $U \cap V = \emptyset$. This contradiction shows that $\tilde{U} \times \tilde{V}$ does not intersect R. Hence R is closed.

Conversely, suppose that R is closed. Given any distinct pair of points $\pi(x)$ and $\pi(y)$ in $X/{\sim}$, there is an open set of the form $\tilde{U} \times \tilde{V}$ containing (x,y) and having no points in R. It follows that $U := \pi(\tilde{U})$ and $V := \pi(\tilde{V})$ are disjoint. Exer. 31 and hypothesis imply that U and V are open. Thus $X/{\sim}$ is hausdorff. ☐

Example 33. We show a typical application of the last result by proving that $\mathbb{P}^n(\mathbb{R})$ is hausdorff. Let $X = \mathbb{R}^{n+1} \setminus \{0\}$. Let the equivalence be as in (e) of Exer. 15. We first show that $\pi \colon X \to \mathbb{P}^n(\mathbb{R})$is open. If $\alpha \in \mathbb{R}$ is nonzero, the map $\varphi_\alpha \colon X \to X$ given be $\varphi_\alpha(x) = \alpha x$ is a homeomorphism. If $U \subset X$ is open, then $[U] = \cup \varphi_\alpha(U)$ where the union is taken over all nonzero reals. Since each $\varphi_\alpha(U)$ is open, their union $[U]$ is also open. By Exer. 31, π is open.

We now show that $\mathbb{P}^n(\mathbb{R})$ is hausdorff. Consider the function $f \colon X \times X \to \mathbb{R}$ is given by $f(x,y) := \sum_{i \neq j} (x_i y_j - x_j y_i)^2$. Then f is continuous and vanishes iff $y = \alpha x$ for some nonzero real α, that is, iff $x \sim y$. Thus $R = \{(x,y) : x \sim y\} = f^{-1}(0)$ is a closed subset of $X \times X$. By Prop. 32, $\mathbb{P}^n(\mathbb{R})$ is hausdorff.

Collapsing and Attaching

We now discuss two construction which are very important Algebraic Topology and which arise as quotient spaces.

Example 34. Let A be a nonempty subset of a topological space. Define $\sim$ to be the equivalence relation on X that identifies all points of A with each other:

$$x \sim y \iff (x = y) \text{ or } (x \in A \text{ and } y \in A).$$

The points of the quotient set $X/\sim$ are the singletons $\{x\}$ for $x \notin A$ and the distinguished point A. The quotient space is most often denoted by X/A. One says that it is obtained by *collapsing* A to a single point.

Let $\pi\colon X \to X/A$ be the quotient map. Then π maps the open subspace $x \setminus A$ of X onto the complement $(X \setminus A) \setminus \{A\}$ of the special point A of X/A. Thus the complement of A in X/A is open, as its inverse image under π is $X \setminus A$.

In fact, π induces a homeomorphism $X \setminus A \cong (X/A) \setminus \{A\}$. Note that the continuous map π is one-one on $X \setminus A$. If $U \subset (X \setminus A)$ is open in $X \setminus A$, then $\pi^{-1}(\pi(U)) = U$ is open in X. Therefore, $\pi(U)$ is open in X/A. We thus see that X/A contains a point whose complement is the same as $X \setminus A$ topologically.

We now consider a specific example. Let $X = B[0,1] \subset \mathbb{R}^2$, the closed unit disc in the plane and $A = S^1$, its boundary. Can you imagine what the quotient space look like? Imagine a circular piece of rubber with a drawstring along its boundary. When the string is drawn tight, a kind of spherical bag results. Therefore, we should expect $X/A \cong S^2$, the unit sphere in $\mathbb{R}^3$.

How do we prove this rigorously? Let p be the north pole of S^2. Geometrically thinking, can we find a map $f\colon X \to S^2$ that sends each point of S^1 to p and maps $X \setminus S^2$ injectively onto $S^2 \setminus \{p\}$? The induced map on X/A would be as required. Look at the closed unit disk $D := \{(x,y,z) \in \mathbb{R}^2 : x^2 + y^2 \leq \pi, z = -1\}$ in the plane tangent to S^2 at the south pole $-p$. We wrap it S^2 by wrapping each radial line segment in this disk onto a meridian of S^2. If you still remember cylindrical and spherical coordinates, what we plan to do is to send a point $P \in D$ with cylindrical coordinates $(r, \theta, -1)$ to a point on S^2 whose spherical coordinates are $(1, \pi - r, \theta)$. Thus f will be the composite of two maps: one is the homeomorphism $(x, y) \mapsto (\pi x, \pi y, -1)$ of X onto D and the second is as described earlier. Since the resulting map is from a compact space to a Hausdorff space, it is closed. It is clearly continuous.

Example 35. Let X and Y be topological spaces and $A \subset X$ be nonempty and closed. Let $f\colon A \to Y$ continuous. Imagine joining X and Y together by gluing each point $a \in A$ to $f(a) \in Y$. The resulting space should be a topological space Z which contains (homeomorphic) copies of $X \setminus A$ and Y in which each $y \in f(A)$ represents an identification of all $a \in f^{-1}(y)$ with y. Our aim is to construct such a space.

Before doing this, we need the notion of sum of two topological spaces. Consider two topological spaces X and Y. Assume that as sets they are disjoint. Consider the union $Z = X \cup Y$. We wish to endow a topology on Z in an obvious manner: call $W \subset Z$ open iff $W \cap X$ and $W \cap Y$ are open in X and Y respectively. Note that this is the same as saying any open set $W \subset Z$ is of the form $W = U \cup V$ with U and V being open in X and Y respectively.

One easily shows that this defines a topology on Z in such a way that both X and Y are open subspaces in Z. The space Z is called the *topological sum* of X and Y

If X and Y are not disjoint, we resort to a standard trick. In stead of X and Y, we consider $X_1 := X \times \{1\}$ and $Y := Y \times \{2\}$. The open sets of X_1 are $U \times \{1\}$. Similarly, we define open sets in Y_1. Clearly, $X_1 \cong X$ and $Y_1 \cong Y$. Let Z be the topological sum of X_1 and Y_1. The space Z is denoted as $X_1 + Y_1$. Via the maps $x \mapsto (x, 1)$ and $y \mapsto (y, 2)$, we see that X and Y are open subsets of Z.

Let us now return to our original notation. Let us first assume that X and Y are disjoint. Then the space $X + Y$ contains both X and Y as open, closed subspaces. We define an equivalence relation $\sim$ on $X + Y$ as follows:

$$\begin{aligned} u \sim v \quad \Longleftrightarrow \quad & (u = v) \text{ or } (u \in A \;\&\; v = f(u)) \text{ or } (v \in A \;\&\; u = f(v)) \\ & \text{or } (u \in A \;\&\; v \in A \;\&\; f(u) = f(v)). \end{aligned}$$

We let $Z = (X + Y)/\sim$. The standard notation for Z is $X \cup_f Y$. We say that $X \cup_f Y$ is obtained by *attaching* X to Y by f. The map f is called the attaching map.

We now show that $X \cup_f Y$ has the desired properties. Let $\pi\colon X + Y \to X \cup_f Y$ be the quotient map. Clearly, $\pi(a) = \pi(f(a))$ for any $a \in A$. Also, we have

$$X \cup_f Y = \pi(X) \cup \pi(Y) = \pi(X \setminus A) \cup \pi(A) \cup \pi(Y),$$

as well as

$$\pi(X) \cap \pi(Y) = \pi(A) = \pi(f(A)).$$

We make the following claims: (i) $\pi(Y)$ is closed in $X \cup_f Y$ and π maps Y homeomorphically onto $\pi(Y)$, (ii) $\pi(X \setminus A)$ is open in $X \cup_f Y$ and π maps $X \setminus A$ homeomorphically onto $\pi(X \setminus A)$.

Proof of Claim (i): The set $\pi(Y)$ is closed since $\pi^{-1}(\pi(Y)) = A \cup Y$ is closed in $X + Y$. The restriction of π to Y is continuous and injective. If F is closed in Y, then $\pi(F)$ is closed in $\pi(Y)$ since $\pi^{-1}(\pi(F)) = A \cup F$ is closed in $X + Y$. This proves (i). The proof of (ii) is similar.

We now take up a specific example. Let $X = [0, 1]$ and $A = \{0, 1\}$. Let $Y = B[0, 1]$, the closed unit disk in $\mathbb{R}^2$. Let $f\colon A \to Y$ be such that $f(0) = f(1) = 0 \in Y$. Can you imagine the space $X \cup_f Y$? It looks line a circle touching a unit disk tangentially at the origin of the disk. More precisely, we show that $X \cup_f Y$ is homeomorphic to the subspace

$$Z = \{(x, y, z) \in \mathbb{R}^3 : z = 0, x^2 + y^2 \leq 1\} \cup \{(x, y, z) \in \mathbb{R}^3 : x = 0, y^2 + (z - 1)^2 = 1\}.$$

See Figure??? To complete the proof, consider the map $g\colon X + Y \to Z$ given by

$$\begin{aligned} g(x) &= (0, \sin 2\pi x, 1 - \cos 2\pi x), \text{ for } x \in X, \\ g(y) &= (y_1, y_2, 0), \text{ for } y \in Y. \end{aligned}$$

The general case is done considering X_1 and Y_1 as earlier and form their sum. Let $h\colon X \to X_1 + Y_1$ and $k\colon Y \to X_1 + Y_1$ be the imbeddings defined earlier. Define an equivalence relation on $X_1 + Y_1$ using these imbeddings and so on. The details are left to the reader.