Raleigh Quotients and Spectral Theorems

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Ex. 1. Let $A\colon \mathbb{R}^n \to \mathbb{R}^n$ be linear. Let $f(v) := \langle Av, v\rangle$ for $v \in \mathbb{R}^n$. Then f is differentiable and $f'(v)(h) = \langle Av, h\rangle + \langle A^*v, h\rangle$. *Hint:* Consider

$$f(v+h) - f(v) = \langle Av, h\rangle + \langle A^*v, h\rangle + 2\langle Ah, h\rangle .$$

Use continuity of A to conclude that

$$\frac{f(v+h) - f(v) - (\langle Av, h\rangle + \langle A^*v, h\rangle)}{\|h\|} \to 0$$

as $\|h\| \to 0$.

Definition 2. Let $v \in \mathbb{R}^n$ be nonzero. The *Rayleigh quotient* of v w.r.t. A is

$$R(v) := \frac{\langle Av, v\rangle}{\langle v, v\rangle}.$$

Note that $R(v) = R(tv)$ for any $0 \neq t \in \mathbb{R}$. Hence we may consider R as a function on the sphere $S := \{u \in \mathbb{R}^n : \|u\| = 1\}$.

Ex. 3. R attains maximum and minimum values on $\mathbb{R}^n \setminus \{0\}$.

Ex. 4. R is differentiable and we have

$$R'(v)(h) = \frac{\langle Av + A^*v - 2R(v)v, h\rangle}{\langle v, v\rangle}.$$

Hint: Note that R can be thought of f/g where $f(v) := \langle Av, v\rangle$ and $g(v) := \langle v, v\rangle$. Apply Ex. 1 and the quotient rule.

Ex. 5. Let A be symmetric: $\langle Av, w\rangle = \langle v, Aw\rangle$. Let R be its Rayleigh quotient. Let v be a critical point of R. (Why does it exist?) Then v is an eigen vector of A. Conversely, if v is a nonzero eigen vector of A, then v is a critical point of R. Deduce the spectral theorem for symmetric maps: There exists an orthonormal basis consisiting og eigen vectors. *Hint:* Note that $R'(v) = 0$ iff $2Av = R(v)v$ and that $(\mathbb{R}v)^\perp$ is invariant under A. Apply induction.

Ex. 6. Let A be orthogonal. Then A has at least one invariant subspace of dimension 1 or 2. *Hint:* If v is a critical point of R, then $Av + A^*v - 2R(v)v = 0$ so that the three vectors Av, $A^*v = A^{-1}v$ and $R(v)v$ are linearly dependent.

Ex. 7. Let A be orthogonal and $V \subset \mathbb{R}^n$ be an invariant vector subspace. Then $V^\perp$ is also A invariant.

Ex. 8. Show that the only orthogonal linear maps of $\mathbb{R}$ are the identity map and the negative of the identity map.

Definition 9. Recall the classification of orthogonal linear maps of $\mathbb{R}^2$: they are either rotations or reflections across a line through the origin. If A is orthogonal on $\mathbb{R}^n$ and if $P \subset \mathbb{R}^n$ is a two dimensional vector subspace invariant under A, then we say it is a rotation (resp. reflection) plane according as whether the restriction of A to P is a rotation or a reflection.

Theorem 10. *If $n \geq 3$ and $A \colon \mathbb{R}^n \to \mathbb{R}^n$ is orthogonal, then A has a rotation plane.*

Proof. Let $n = 3$. Then there exists a two dimensional vector subspace V invariant under A (by Ex. 6 and Ex. 7). If the restriction of A is not a rotation, then V has a one dimensional subspace L w.r.t. which $A\mid_V$ is a reflection. If $V^\perp$ denotes the one dimensional orthogonal complement of L in $\mathbb{R}^3$, then $A\mid_{L^\perp}$ is either the identity or a reflection w.r.t. the origin. (See Ex. 8.) In the first case, span $\{L \cup V^\perp\}$ is rotation plane.

Let $n = 4$. Let V be an invariant vector subspace of A. If $\dim V = 1$, then $\dim V^\perp = 3$ and by the last paragraph, we are through. If $\dim V = 2$, and if $A\mid_V$ is a rotation we are through. If both V and $V^\perp$ are reflection planes, then span $\{L \cup M\}$ is a rotation plane, where L (resp. M) is the line of reflection.

Now the proof is completed by induction. □

Ex. 11. Let $A \colon \mathbb{R}^n \to \mathbb{R}^n$ be orthogonal. Then

(a) If $n = 2k + 1$, then $\mathbb{R}^n$ is the orthogonal direct sum of k rotation planes and an invariant line.

(b) If $n = 2(k + 1)$, then $\mathbb{R}^n$ is the orthogonal direct sum of k rotation planes and an invariant plane.

Hint: Induction.

Ex. 12. Any orthogonal linear transformation of $\mathbb{R}^n$ can be expressed as a composition of atmost n reflections. *Hint:* Induction.